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THREE-PHASE HEAT TRANSFER:  
TRANSIENT CONDENSING WITH FREEZING  
ON A NONISOTHERMAL INCLINED PLATE

*by William A. Olsen, Jr., and Frank B. Molls*

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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## ABSTRACT

The transient phase change process that occurs when a vapor contacts a very cold inclined plate is analyzed. The vapor condenses to form a growing liquid layer that flows down the plate. Below this two-phase region there is a three-phase region where the thickening condensate layer flows over a growing frozen layer. Heat and mass are transferred across moving phase boundaries. The solution involves the Karmen-Pohlhausen method which results in quasi-linear partial differential equations for the phase thicknesses that are solved by characteristic and finite difference methods. Numerous two-phase special cases are also discussed.

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# THREE-PHASE HEAT TRANSFER: TRANSIENT CONDENSING WITH FREEZING ON A NONISOTHERMAL INCLINED PLATE

by William A. Olsen, Jr., and Frank B. Molls

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## SUMMARY

The subject of this analytical study is transient condensing with freezing of a slowly moving saturated vapor upon an inclined plate. The plate has no thermal capacity and is cooled by a fluid that is below the freezing point of the vapor. At the top of the plate, there is a two-phase region where a growing condensate liquid layer flows (laminar flow) down the plate. Below that region, there is a three-phase region where the growing condensate layer flows over a thickening frozen layer.

The solution to the problem employs the Karman-Pohlhausen integral method with assumed linear temperature profiles across the phase layers. This method reduces the problem to quasi-linear first-order hyperbolic partial differential equations, with the phase thicknesses as dependent variables. For the two-phase region, the solution involves a single characteristic curve, which is represented by algebraic equations. Families of characteristic curves, described by a network of ordinary differential equations, are involved in the solution for the three-phase region. The hyperbolic partial differential equations for the three-phase region are also solved by a direct finite difference equation approximation for comparison purposes.

Results of the analysis indicate that freezing will start at and beyond a fixed point, as soon as the insulating condensate layer is thick enough for the wall temperature at that point to drop to the freezing temperature. The solid and condensate layers, which have a shape similar to boundary layers, grow until they attain a steady-state profile.

This analysis specializes to a number of cases which are discussed herein. Some of these cases are not covered in the literature.

## INTRODUCTION

This report deals with the transient and steady-state phase change processes that occur when a vapor suddenly comes in contact with an inclined nonisothermal plate. The plate is cooled by a fluid that is below the freezing point of the vapor. In one area of the plate condensing occurs, while in another area condensing accompanied by freezing occurs. The literature for two-phase problems (e.g., freezing, melting, or condensing) is extensive and is summarized in references 1 and 2. However, the three-phase heat-transfer problem, which combines two-phase problems, has apparently received little attention. There has been little interest in the three-phase problem because it occurs only rarely in nature, and because commercial applications have been lacking. The reason for this is that large temperature differences are normally required in order to span across the thermodynamic phase space from the vapor to the solid region by crossing the liquid region. Now however, large temperature differences can often occur because of the cold environments of space missions, and because cryogenic liquids and liquid metals are broadly applied. A major problem of heat-transfer systems that operate in an environment colder than the freezing point of the vapor is that a solid layer could occur in the vapor flow passages. Any such solid layer that forms there would usually be detrimental in that it restricts flow or reduces the heat transfer.

In reference 3 the transient horizontal plate and steady-state inclined plate cases of the three-phase problem were considered. The problem considered herein is an extension to the work of reference 3. This extension considers transient condensing and freezing of a slowly moving vapor upon a nonisothermal inclined plate. The condensate is subjected to a constant body force (e.g., an inclined plate in a constant gravity field). The plate shown in figure 1 is cooled by a coolant of constant temperature and constant heat-transfer coefficient (e.g., a well-mixed coolant bath). Initially the plate is dry, and suddenly conditions change such that condensing from the vapor commences. This may later be followed by freezing of the condensate liquid beyond some point on the plate ( $X_s$  on fig. 1). Before steady-state phase thicknesses are attained, there is a growing condensate layer (two-phase region) above  $X_s$ . Below  $X_s$  there is a three-phase region, where the condensate layer flows over a solid layer that grows by freezing some of the condensate liquid. Heat and mass are transferred across the moving phase boundaries in both the two- and three-phase regions.

In order to continue the description of the physical model, it is necessary to mention two effects of a noncondensable gas. The experiment of reference 3 dealt with condensing and freezing of water vapor, in the presence of some air, upon a liquid-nitrogen-cooled vertical cold plate. These experimental results qualitatively indicated that a noncondensable gas (air) has a large effect upon the manner of formation and the type of solid layer formed from the vapor (water vapor). As the noncondensable gas

fraction increased, the type of solid formed from the vapor changed from an ice-like solid to an ice-and-frost-laminated composite solid, and finally to frost alone at large air fractions. Only the first case, condensing from a pure vapor where a solid (ice-like) layer is formed, is considered in the transient case of this report. The second effect is that a small amount of noncondensable gas (small enough for an ice-like solid) offers considerable thermal resistance to heat transfer such that a much large solid layer grows than with a pure vapor. This second effect is quantitatively considered in appendix C by a modification to the limiting case of steady state.

An exact solution (i. e., algebraic equations) to this problem is unlikely. The approximate Karman-Pohlhausen integral method was recommended for three-phase problems in reference 3, where many methods were compared. Accordingly, this is used herein and results in a set of quasi-linear first-order partial differential equations to solve. Both characteristics and finite difference techniques are employed in their solution. The accuracy of these methods is compared to the exact solutions of special cases.

The analysis of transient condensing with freezing specializes to a number of cases, some of which have not been reported in the literature. Transient condensing and freezing on a horizontal plate and steady-state condensing and freezing on an inclined nonisothermal plate are two such cases. Transient condensing on a nonisothermal plate is another. With minor changes in the initial conditions, the analysis can handle transient condensing with melting of a thin slab of solid by its vapor; and, with a minor modification, transient freezing of a liquid flowing past a cold plate can be analyzed.

## DERIVATION OF DIFFERENTIAL EQUATIONS

Consider the plate of no thermal capacity (i. e., a thin metal plate a short time after the start of the transient) shown in figure 1. One side of the plate is in contact with a coolant of constant heat-transfer coefficient  $h_c$  and constant temperature  $T_c$ . Suddenly, a slowly moving vapor comes in uniform contact with the opposite side of the cold plate and the vapor starts to condense while the pressure  $P$  is held constant (fig. 1(a)). The condensate liquid layer will grow thicker while it flows down the plate (fig. 1(b)). The flow is caused by some steady body force (e. g., gravity with the plate inclined from the horizontal) and/or vapor drag caused by momentum transfer and friction. As the insulating layer of condensate liquid thickens, the temperature of the plate surface in contact with the phases  $T_w$  will fall. The coolant temperature  $T_c$  may be below the freezing temperature of the vapor  $T_{SL}$ . If it is,  $T_w$  may fall to the freezing temperature  $T_{SL}$ . When this happens, a solidified layer will start to grow at those areas of the plate. Further growth of the condensate liquid will result in a corresponding growth in

the solidified layer until a steady state is attained (fig. 1(c)). Because of the finite thermal resistance of the coolant and plate  $b$ , the start of freezing will occur some distance,  $x = X_s(t)$ , below the start of the cold plate,  $x = 0$ . This problem, therefore, consists of two regions: one where transient condensing (two phases) occurs for  $x \leq X_s$ , and another where transient condensing and freezing (three phases) occur for  $x > X_s$ . Should the coolant and plate thermal resistance  $b$  be zero, the plate would be at a constant wall temperature,  $T_w = T_c$ , and freezing would start at  $x = 0$ .

This three-phase problem with its moving phase boundaries can be quite complicated, such that a number of simplifying assumptions that do not greatly compromise the physics of the problem are necessary. The three-phase problem is essentially a combination of the two-phase problems of condensing and freezing. The literature for each of these two-phase studies is extensive. These two-phase studies are used herein to supply physically reasonable simplifying assumptions for the three-phase problem.

Consider the condensate liquid layer first. The literature for steady-state condensing on a vertical isothermal plate is well summarized in references 2 and 4, and transient condensing on a vertical isothermal plate has been studied in reference 5. These references indicate that the physical assumptions necessary to obtain the equations of Nusselt's original simple model for condensing on a plate, as described in reference 6, are adequate for condensing of a pure slowly moving vapor. Furthermore, the Nusselt model can be adapted to successfully describe the effect of noncondensable gas and also liquid-metal condensing. Based upon the simple Nusselt model, the following assumptions are made for the analysis of the condensate layer in this report: no waves, small phase and wall boundary curvature, constant properties, axial gradients are much less than normal gradients and axial velocities are much larger than normal velocities (boundary layer assumptions), saturated pure vapor at constant pressure, laminar flow, and, finally, the momentum convection terms (e.g.,  $\partial u_l / \partial t + u_l \partial u_l / \partial x + v_l \partial u_l / \partial y$ ) have a negligible effect upon the flow in the thin condensate layer. Unlike the Nusselt model, the condensate energy convection terms are included herein; and the wall temperature is allowed to vary, but axial heat transfer is neglected. The coolant temperature and the thermal resistance of the coolant and plate are assumed to be constant.

By these assumptions, the general continuity, momentum, and energy equations that are summarized in reference 7 are simplified to the following equations that describe the condensate liquid layer. The coordinate system is fixed to the plate, as shown in figure 1. Continuity is described by

$$\frac{\partial \rho_l}{\partial t} + \frac{\partial}{\partial x} (\rho_l u_l) + \frac{\partial}{\partial y} (\rho_l v_l) = 0 \quad (1)$$

where the constant density assumption has not as yet been applied to equation (1) for convenience. The momentum equation for the thin slowly moving condensate layer simplifies greatly and involves only a bouyant force term and a viscous drag term.

$$(\rho_l - \rho_v)g_x + \mu_l \frac{\partial^2 u_l}{\partial y^2} = 0 \quad (2)$$

The energy equation for the condensate layer is given by equation (3).

$$\rho_l c_l \left( \frac{\partial T_l}{\partial t} + u_l \frac{\partial T_l}{\partial x} + v_l \frac{\partial T_l}{\partial y} \right) = - \frac{\partial q_l}{\partial y} = k_l \frac{\partial^2 T_l}{\partial y^2} \quad (3)$$

Equations (1) to (3) are also used in reference 5 to describe transient condensing on an isothermal vertical plate.

Just as with the condensate layer, it can be expected that the axial gradients in a thin stationary solid layer are small compared to the normal gradients. By making the additional assumptions that heat is transfered in the solid solely by conduction and that the solid properties are constant, the following equation can be written to describe heat transfer in that layer:

$$\rho_s c_s \frac{\partial T_s}{\partial t} = - \frac{\partial q_s}{\partial y} = k_s \frac{\partial^2 T_s}{\partial y^2} \quad (4)$$

Reference 1 used this equation for one-dimensional transient freezing on a plate.

The boundary conditions of the variables of the problem (e. g. ,  $T(x, y, t)$  ,  $Y_{LV}(x, t)$  ,  $q(x, y, t)$  ,  $u(x, y, t)$  , etc.) are listed belcw. The term  $T(x, y, t)$  means that the temperature is a function of  $x$ ,  $y$ , and  $t$ .

(1) Initially, the slowly moving saturated vapor is in uniform contact with a cold dry plate so that for all  $x$  at  $t = 0$  the following are the boundary conditions:

$$Y_{LV}(x, t=0) = 0 \quad (5)$$

$$Y_{SL}(x, 0) = 0 \quad (6)$$

$$T_w(x, 0) = T_c \quad (7)$$

(2) At the leading edge of the cold plate ( $x = 0$ ), the condensate thickness is zero.

$$Y_{LV}(0, t) = 0 \quad (8)$$

(3) Along the inner liquid condensate layer boundary (fig. 1), there are two regions where the boundary conditions differ.

(a) Along the plate ( $y = 0$ ) where there is no solid layer ( $x \leq X_s(t)$ ), the following conditions occur:

$$u_l(x, y=0, t) = 0 \quad (9)$$

$$T_l(x, 0, t) = T_w(x, t) \quad (10)$$

$$-q_l(x, 0, t) = k_l \frac{\partial T_l}{\partial y}(x, 0, t) = \frac{T_w(x, t) - T_c}{b} = \left(\frac{Q}{A}\right)_c(x, t) \quad (11)$$

where  $b$  is the combined coolant and plate thermal resistance,  $b = 1/h_c + d_m/k_m$ .

(b) At the point where freezing starts on the plate ( $x = X_s(t)$ ), the wall temperature is at the freezing point.

$$Y_{SL}(x=X_s(t), t) = 0 \quad (12)$$

$$T_w(X_s, t) = T_{SL} \quad (13)$$

(c) The boundary conditions along  $y = Y_{SL}(x, t)$  (the solid-liquid interface), which apply where there is a solid layer at  $x > X_s(t)$ , are given by

$$\begin{aligned} & -q_s(x, y=Y_{SL}, t) + q_l(x, Y_{SL}, t) \\ & = k_s \frac{\partial T_s}{\partial y}(x, Y_{SL}, t) - k_l \frac{\partial T_l}{\partial y}(x, Y_{SL}, t) \\ & = -\left(\frac{m}{A}\right)_{SL} L_{SL} = \left(\frac{m}{A}\right)_{fr} L_{SL} \end{aligned} \quad (14)$$

$$u_l \langle x, Y_{SL}, t \rangle = 0 \quad (15)$$

$$T_l \langle x, Y_{SL}, t \rangle = T_s \langle x, Y_{SL}, t \rangle = T_{SL} \quad (16), (17)$$

(4) Along the plate ( $y = 0$ ) where there is a solid layer ( $x > X_s \langle t \rangle$ ), the following conditions are used:

$$T_s \langle x, y=0, t \rangle = T_w \langle x, t \rangle \quad (18)$$

$$-q_s \langle x, 0, t \rangle = k_s \frac{\partial T_s}{\partial y} \langle x, 0, t \rangle = \frac{T_w \langle x, t \rangle - T_c}{b} = \left( \frac{Q}{A} \right)_c \langle x, t \rangle \quad (19)$$

(5) Along the entire liquid-vapor interface ( $x > 0$ ) which follows  $y = Y_{LV} \langle x, t \rangle$ , the boundary conditions are

$$T_l \langle x, Y_{LV}, t \rangle = T_{LV} \langle P \rangle \quad (20)$$

$$-q_l \langle x, Y_{LV}, t \rangle = k_l \frac{\partial T_l}{\partial y} \langle x, Y_{LV}, t \rangle = - \left( \frac{m}{A} \right)_{LV} L_{LV} = \left( \frac{m}{A} \right)_{cd} L_{LV} \quad (21)$$

where the vapor was assumed to be saturated at constant pressure so that  $\partial T_v / \partial y \langle x, Y_{LV}, t \rangle = 0$ . The basis for equations (13), (14), (16), (17), (20), and (21) is discussed in appendix D of reference 3 and in reference 8. The final boundary condition is the shear stress at  $y = Y_{SL} \langle x, t \rangle$ . Reference 2, where this boundary condition is discussed in detail, indicates that this shear stress is composed of contributions from the momentum transfer of condensation and form drag. Unfortunately, even low vapor velocities can readily cause considerable waviness of the liquid-vapor interface, which appreciably affects heat transfer. This situation adds too much complication to be considered herein; therefore, it is assumed that the vapor velocity is low enough (e. g.,  $U_v < 10$  ft/sec (3 m/sec)) that the effect of vapor drag can be neglected. The effect of the small ripple waves caused by gravity is also neglected. The resulting shear stress boundary condition is given by equation (22).

$$\frac{\partial u_l}{\partial y} \langle x, Y_{LV}, t \rangle = 0 \quad (22)$$

This analysis is therefore realistic for external condensing (e. g., a plate condenser)

but not for internal condensing (e.g., condensing inside a tube), where the vapor velocity is very high.

The next task is to apply the Karman-Pohlhausen integral method to the differential equations (eq. (1) to (4)) by first integrating them over the time- and position-varying phase thicknesses for a  $dx$  element. Following this operation, temperature profiles are assumed and the boundary conditions applied. These operations will result in a simpler system of partial differential equations, where the phase thicknesses are the dependent variables.

A mass balance on the liquid condensate layer is obtained from the continuity equation (eq. (1)) by this method (see appendix D in ref. 3 for more detail). Wherever there is a solid layer ( $x > X_s(t)$ ), these operations result in the following integro-differential equation.

$$\rho_l \frac{\partial}{\partial x} \int_{Y_{SL}(x,t)}^{Y_{LV}(x,t)} u_l dy + \rho_l \left( \frac{\partial Y_{LV}}{\partial t} - \frac{\partial Y_{SL}}{\partial t} \right) = - \left( \frac{m}{A} \right)_{LV} + \left( \frac{m}{A} \right)_{SL} = \left( \frac{m}{A} \right)_{cd} - \left( \frac{m}{A} \right)_{fr} \quad (23a)$$

where the constant density assumption has finally been applied. Wherever there is no solid layer ( $x \leq X_s(t)$ ), continuity of mass becomes

$$\rho_l \frac{\partial}{\partial x} \int_0^{Y_{LV}(x,t)} u_l dy + \rho_l \frac{\partial Y_{LV}}{\partial t} = - \left( \frac{m}{A} \right)_{LV} = \left( \frac{m}{A} \right)_{cd} \quad (23b)$$

The mass balance on the solid layer is similarly derived.

$$\rho_s \frac{\partial Y_{SL}}{\partial t} = - \left( \frac{m}{A} \right)_{SL} = \left( \frac{m}{A} \right)_{fr} \quad (24)$$

The energy equation for the condensate layer (eq. (3)) is similarly integrated. Wherever there is a solid layer ( $x > X_s(t)$ ), equation (25) describes the energy balance.

$$\begin{aligned}
\rho_l c_l \frac{\partial}{\partial t} \int_{Y_{SL}\langle x, t \rangle}^{Y_{LV}\langle x, t \rangle} (T_l - T_{LV}) dy + \rho_l c_l \frac{\partial}{\partial x} \int_{Y_{SL}\langle x, t \rangle}^{Y_{LV}\langle x, t \rangle} u_l (T_l - T_{LV}) dy \\
+ c_l (T_{LV} - T_{SL}) \left( \frac{m}{A} \right)_{SL} = -q_l \langle x, Y_{LV}, t \rangle + q_l \langle x, Y_{SL}, t \rangle
\end{aligned} \quad (25)$$

Equation (26) describes the energy balance on the condensate layer where there is no solid layer at  $x \leq X_s \langle t \rangle$ .

$$\begin{aligned}
\rho_l c_l \frac{\partial}{\partial t} \int_0^{Y_{LV}\langle x, t \rangle} (T_l - T_{LV}) dy + \rho_l c_l \frac{\partial}{\partial x} \int_0^{Y_{LV}\langle x, t \rangle} u_l (T_l - T_{LV}) dy = -q_l \langle x, Y_{LV}, t \rangle \\
+ q_l \langle x, 0, t \rangle
\end{aligned} \quad (26)$$

Integrating equation (4) over the solid layer thickness results in the following equation for the solid layer:

$$\rho_s c_s \frac{\partial}{\partial t} \int_0^{Y_{SL}\langle x, t \rangle} (T_s - T_{SL}) dy = -q_s \langle x, Y_{SL}, t \rangle + q_s \langle x, 0, t \rangle \quad (27)$$

The integration of the momentum equation (eq. (2)) is considered next. Because of the no-vapor-shear assumption, this equation can be very simply integrated. By using the velocity boundary conditions (eqs. (15) and (27)), the following velocity profile is obtained for where there is a solid layer at  $x > X_s \langle t \rangle$ .

$$u_l = u_l \langle y \rangle = n \left[ \frac{Y_{SL}^2}{2} - \frac{y^2}{2} + Y_{LV}(y - Y_{SL}) \right] \quad (28a)$$

And wherever there is no solid layer,  $x \leq X_s \langle t \rangle$

$$u_l = n \left( -\frac{y^2}{2} + y Y_{LV} \right) \quad (28b)$$

where

$$n = \frac{(\rho_l - \rho_v)g_x}{\mu_l} \quad (28c)$$

This velocity profile is now used in the mass and energy balances (eqs. (23), (25), and (26)) along with assumed temperature profiles across the phase layers in order to complete the indicated integrations.

References 1 and 4 have shown that linear temperature profiles are good approximations for the temperature distribution across thin phase layers where there is a phase change. With this physical insight, the following linear temperature profiles are assumed, which satisfy the temperature boundary conditions (e.g., eqs. (10), (16) to (18), and (20)). For the condensate layer where there is freezing ( $x > X_s(t)$ ), assume the following linear temperature profile:

$$T_l - T_{LV} \equiv (T_{LV} - T_{SL}) \left( \frac{y - Y_{SL}}{Y_{LV} - Y_{SL}} - 1 \right) \quad (29)$$

For the condensate layer where there is no freezing ( $x \leq X_s(t)$ ), the following linear relation is assumed:

$$T_l - T_{LV} \equiv (T_{LV} - T_w(x, t)) \left( \frac{y - Y_{LV}}{Y_{LV}} \right) \quad (30)$$

And for the solid layer at  $x > X_s(t)$

$$T_s - T_{SL} \equiv (T_{SL} - T_w(x, t)) \left( \frac{y - Y_{SL}}{Y_{SL}} \right) \quad (31)$$

Equations (30) and (31) involve a variable wall temperature, which is removed from the temperature profiles by using the boundary conditions. For  $x \leq X_s(t)$ , substitute equation (30) into equation (11) and solve for  $T_w(x, t)$ .

$$T_w \langle x, t \rangle = \frac{bT_{LV} + \frac{Y_{LV}}{k_l} T_c}{\frac{Y_{LV}}{k_l} + b} \quad (32)$$

for  $x > X_s \langle t \rangle$ , substitute equation (31) into equation (19).

$$T_w \langle x, t \rangle = \frac{bT_{SL} + \frac{Y_{SL}}{k_s} T_c}{\frac{Y_{SL}}{k_s} + b} \quad (33)$$

These relations are used below to eliminate  $T_w \langle x, t \rangle$  when the integro-differential equations are integrated and put into a form that contains only the phase thicknesses as variables.

The heat flux to the coolant  $(Q/A)_c$  from the condensate layer at  $x \leq X_s \langle t \rangle$  is determined by substituting equation (32) into equation (11).

$$\left( \frac{Q}{A} \right)_c \langle x, t \rangle = -q_l \langle x, 0, t \rangle = \frac{T_{LV} - T_c}{\frac{Y_{LV}}{k_l} + b} \quad (34)$$

The heat flux to the coolant from the solid layer at  $x > X_s \langle t \rangle$  is determined from equations (33) and (19).

$$\left( \frac{Q}{A} \right)_c \langle x, t \rangle = -q_s \langle x, 0, t \rangle = \frac{T_{SL} - T_c}{\frac{Y_{SL}}{k_s} + b} \quad (35)$$

The heat flux into the solid layer from the condensate layer at  $x > X_s \langle t \rangle$  is determined from equations (29) and (14).

$$-q_s \langle x, Y_{SL}, t \rangle = -q_l \langle x, Y_{SL}, t \rangle - \left( \frac{m}{A} \right)_{SL} L_{SL} = \frac{k_l (T_{LV} - T_{SL})}{Y_{LV} - Y_{SL}} - \left( \frac{m}{A} \right)_{SL} L_{SL} \quad (36)$$

And finally the heat flux into the liquid for all  $x$  is given by equation (21).

The next steps in the analysis involve the evaluation of the integrals of equations (23) and (25) to (27) by using the velocity profiles, temperature profiles, and heat flux boundary conditions that have just been derived. These evaluations are greatly simplified by changing the  $Y_{LV}$  and  $Y_{SL}$  variables to the phase thickness variables  $\delta$  and  $\Delta$ . Where there is no solid layer ( $x \leq X_s \langle t \rangle$ )

$$Y_{LV} \langle x, t \rangle \equiv \delta \langle x, t \rangle = \delta \quad (37)$$

while for  $x > X_s \langle t \rangle$

$$Y_{LV} \langle x, t \rangle - Y_{SL} \langle x, t \rangle \equiv \delta \langle x, t \rangle = \delta \quad (38a)$$

and

$$Y_{SL} \langle x, t \rangle \equiv \Delta \langle x, t \rangle = \Delta \quad (38b)$$

Substitute the velocity profile (eq. (28a)) into equation (23a) and perform the indicated integrations. Then use equation (37) to obtain, for  $x > X_s \langle t \rangle$ ,

$$\rho_l \frac{\partial \delta}{\partial t} + \rho_l n \delta^2 \frac{\partial \delta}{\partial x} = - \left( \frac{m}{A} \right)_{LV} + \left( \frac{m}{A} \right)_{SL} \quad (39a)$$

For  $x \leq X_s \langle t \rangle$ , where equations (38a) and (28b) were used in equation (23b), the following mass balance is obtained:

$$\rho_l \frac{\partial \delta}{\partial t} + \rho_l n \delta^2 \frac{\partial \delta}{\partial x} = - \left( \frac{m}{A} \right)_{LV} \quad (39b)$$

Use equation (38b) in equation (24) to obtain a mass balance for the solid layer.

$$\rho_s \frac{\partial \Delta}{\partial t} = - \left( \frac{m}{A} \right)_{SL} \quad (40)$$

Substitute equation (33) into equation (31) to eliminate  $T_w$ , and use this result along

with equation (38b) to evaluate the integral in equation (27). The heat fluxes are evaluated by using equations (35), (36), and (40). This manipulation results in equation (41), which describes the growth of the solid layer at any position  $x$ , where  $x > X_S(t)$ .

$$-\rho_s c_s (T_{SL} - T_c) \frac{\partial}{\partial t} \left( \frac{\Delta^2}{\Delta + k_s b} \right) = \frac{k_l (T_{LV} - T_{SL})}{\delta} + \rho_s L_{SL} \frac{\partial \Delta}{\partial t} - \frac{T_{SL} - T_c}{\frac{\Delta}{k_s} + b} \quad (41)$$

Define a subcooling parameter  $S_1$  according to

$$S_1 \equiv \frac{c_s (T_{SL} - T_c)}{L_{SL}} \quad (42)$$

and rearrange equation (41) to obtain the final form of the solid layer equation.

$$\rho_s L_{SL} \left\{ 1 + \frac{S_1}{2} \left[ 1 - \left( \frac{k_s b}{\Delta + k_s b} \right)^2 \right] \right\} \frac{\partial \Delta}{\partial t} = \frac{k_s (T_{SL} - T_c)}{\Delta + k_s b} - \frac{k_l (T_{LV} - T_{SL})}{\delta} \quad (43)$$

Equation (43) relates the heat of fusion energy generated by solid layer growth (i.e., freezing) and the energy that is required to cool the solid layer below the freezing temperature to the difference between the heat flux transferred to the coolant and the heat flux into the solid layer.

The equation for the condensate layer, where there is no solid layer at  $x \leq X_S(t)$ , is obtained next. First, substitute equation (32) into equation (30) to eliminate  $T_w(x, t)$ , and substitute that result with equations (28b) and (37) into equation (26) in order to evaluate the integrals. Then, substitute equation (39b) into equation (21), and use that result along with equations (34) and (37) to achieve the following result:

$$\begin{aligned} \frac{\rho_l c_l}{2} (T_{LV} - T_c) \frac{\partial}{\partial t} \left( \frac{\delta^2}{\delta + b k_l} \right) + \rho_l L_{LV} \frac{\partial \delta}{\partial t} + \rho_l n L_{LV} \delta^2 \frac{\partial \delta}{\partial x} + \frac{\rho_l c_l n}{8} (T_{LV} - T_c) \frac{\partial}{\partial x} \left( \frac{\delta^4}{\delta + b k_l} \right) \\ = \frac{T_{LV} - T_c}{\left( \frac{\delta}{k_l} \right) + b} \end{aligned} \quad (44)$$

Define a subcooling parameter  $S_3$  as

$$S_3 \equiv \frac{c_l(T_{LV} - T_c)}{L_{LV}} \quad (45)$$

and rearrange equation (44) to obtain the following equation, which describes the growth of the condensate layer where there is no solid layer at  $x \leq X_s(t)$ :

$$\rho_l L_{LV} \left\{ 1 + \frac{S_3}{2} \left[ 1 - \left( \frac{bk_l}{\delta + bk_l} \right)^2 \right] \right\} \frac{\partial \delta}{\partial t} + \rho_l n L_{LV} \delta^2 \frac{\partial \delta}{\partial x} + \frac{\rho_l n L_{LV}}{8} S_3 \frac{\partial}{\partial x} \left( \frac{\delta^4}{\delta + bk_l} \right) = \frac{k_l (T_{LV} - T_c)}{\delta + bk_l} \quad (46)$$

Equation (46) relates the heat of vaporization energy liberated by condensation, the energy required to cool the condensate below saturation, and the thermal energy convected away, to the heat flux energy transferred to the coolant.

The describing equation for the condensate layer, where there is a solid layer at  $x > X_s(t)$ , is determined from equation (25). Use equations (38), (28a), and (29) to evaluate the integrals; then substitute equation (40) into equations (39a) and (36). Now, use these results with equation (38), to evaluate the heat flux terms of equation (25). Substitute these results into equation (25) as indicated and obtain the following equation:

$$\begin{aligned} \frac{\rho_l c_l}{2} (T_{LV} - T_{SL}) \frac{\partial \delta}{\partial t} + \rho_l L_{LV} \frac{\partial \delta}{\partial t} + \rho_s c_l (T_{LV} - T_{SL}) \frac{\partial \Delta}{\partial t} + \rho_s L_{LV} \frac{\partial \Delta}{\partial t} \\ + \frac{3}{8} \rho_l c_l n (T_{LV} - T_{SL}) \delta^2 \frac{\partial \delta}{\partial x} + \rho_l L_{LV} n \delta^2 \frac{\partial \delta}{\partial x} = \frac{k_l (T_{LV} - T_{SL})}{\delta} \end{aligned} \quad (47)$$

Define another subcooling parameter as

$$S_2 \equiv \frac{c_l (T_{LV} - T_{SL})}{L_{LV}} \quad (48)$$

and transform equation (47) into equation (49), which describes the condensate growth where there is a solid layer ( $x > X_s(t)$ ).

$$\rho_l L_{LV} \left( \frac{S_2}{2} + 1 \right) \frac{\partial \delta}{\partial t} + \underbrace{\rho_s L_{LV} (S_2 + 1) \frac{\partial \Delta}{\partial t}}_{\text{Coupling term}} + \rho_l L_{LV} \left( \frac{3}{8} S_2 + 1 \right) \delta^2 \frac{\partial \delta}{\partial x} = \frac{k_l (T_{LV} - T_{SL})}{\delta} \quad (49)$$

Equation (49) can be described the same way as equation (46), with one notable exception. There is a term involving  $\partial \Delta / \partial t$ , which is the additional heat generated by condensing the extra liquid that is frozen to form a layer of solid. This term couples equation (49) to equation (43).

Equation (46) is solved wherever there is no solid layer (i. e.,  $x \leq X_s \langle t \rangle$ ), while equations (43) and (49) are solved simultaneously wherever there is a solid layer (i. e.,  $x > X_s \langle t \rangle$ ). The differential equations describing the problem (i. e., eqs. (46), (43), and (49)) are quasi-linear first-order simultaneous partial differential equations whose solution is considered in the next section.

The solution of the problem also depends upon the point where freezing starts  $X_s \langle t \rangle$ . According to the boundary condition there (eq. (13)) the wall temperature at  $X_s \langle t \rangle$  is at the freezing point.

$$T_w \langle x = X_s, t \rangle = T_{SL} \quad (50)$$

Substituting this fact into equation (32) gives an equation for the condensate thickness  $\delta$  at  $x = X_s \langle t \rangle$ .

$$\delta \langle X_s, t \rangle \equiv \delta_s = \frac{k_l b (T_{LV} - T_{SL})}{T_{SL} - T_c} \quad (51)$$

Notice that  $\delta_s$  is not a function of time or position. It is a constant for this problem since the parameters of equation (51) were assumed to be constant. In order to have freezing (i. e., in order for  $\delta_s \geq 0$ ), the coolant temperature must be less than the freezing temperature ( $T_c \leq T_{SL}$ ). Instead of using  $X_s \langle t \rangle$  as the switchover point between equation (46) and equations (43) and (49), it is easier to solve equation (46) for when  $\delta \leq \delta_s$  and equations (43) and (49) wherever  $\delta > \delta_s$ . Therefore, the boundary conditions used in the two-phase (or condensing with no freezing) region ( $\delta \leq \delta_s$ ) are given by

$$\left. \begin{aligned} \delta \langle x, 0 \rangle &= 0 \\ \delta \langle 0, t \rangle &= 0 \end{aligned} \right\} \quad (52)$$

whereas in the three-phase region, where there is condensing with freezing ( $\delta > \delta_s$ ),

$$\left. \begin{aligned} \delta\langle X_s, t \rangle &= \delta_s \\ \Delta\langle x \geq X_s, t_s \rangle &= 0 \\ \Delta\langle X_s, t \rangle &= 0 \end{aligned} \right\} \quad (53)$$

## METHODS OF SOLUTION

The next task is to solve the partial differential equations that were derived in the previous section. In this section it is shown that the analytical treatment for the transient condensing region ( $x \leq X_s$ ) is quite different from the treatment of the three-phase region ( $x > X_s$ ). The transient condensing region can be solved by a simple exact single characteristic solution. Families of characteristic curves, described by ordinary differential equations, are necessary in the three-phase region. A direct finite difference solution to the partial differential equations is also employed in the three-phase region for comparison. The characteristic solution in the three-phase region (i. e., families of characteristic curves) is somewhat unusual because one characteristic is a zero characteristic. Stability and accuracy requirements necessitated small time and position increments of either the direct finite difference or characteristic solution in the three-phase region, thereby resulting in long computational time. Computing time for a given accuracy was reduced by concentrating the smallest position elements near the start of freezing location  $X_s$ . It turned out that the stability, the accuracy, and the computation time of the characteristic and finite difference methods were essentially equivalent. A single characteristic solution, which does not require lengthy computations, was possible in the three-phase region when the term coupling the two describing differential equations was neglected. This term is zero only at the start of freezing and at steady state, so that an appreciable error might result if it is neglected.

Details of the characteristic solutions are now discussed. This is followed by a detailed discussion of the direct finite difference solution.

# CHARACTERISTIC EQUATIONS METHOD

## Transient Condensing Region

As indicated in reference equation (46), the partial differential equation describing transient condensing for  $x \leq X_s(t)$ , is rewritten in the following standard form.

$$f_1(\delta) \frac{\partial \delta}{\partial t} + f_2(\delta) \frac{\partial \delta}{\partial x} = 1 \quad (54)$$

The result is given by

$$\begin{aligned} \frac{\rho_l L_{LV}}{k_l(T_{LV} - T_c)} & \left( \left\{ \left[ \left( 1 + \frac{S_3}{2} \right) - \frac{S_3}{2} \left( \frac{bk_l}{bk_l + \delta} \right)^2 \right] (bk_l + \delta) \right\} \frac{\partial \delta}{\partial t} \right. \\ & \left. + \left\{ (\delta + bk_l) \left[ n\delta^2 + \frac{S_3 n}{8} \frac{(3\delta + 4bk_l)\delta^3}{(\delta + bk_l)^2} \right] \right\} \frac{\partial \delta}{\partial x} \right) = 1 \end{aligned} \quad (55)$$

This then implies that

$$\frac{dx}{f_2(\delta)} = \frac{dt}{f_1(\delta)} = d\delta \quad (56)$$

so that the single characteristic curve, described by equation (57), exists.

$$\left( \frac{dx}{dt} \right)_c = \frac{f_2(\delta)}{f_1(\delta)} = \frac{n\delta^2 + \frac{S_3 n}{8} \frac{(3\delta + 4bk_l)\delta^3}{(bk_l + \delta)^2}}{\left( 1 + \frac{S_3}{2} \right) - \frac{S_3}{2} \left[ \frac{bk_l}{bk_l + \delta} \right]^2} \quad (57)$$

Assume that  $S_3 \equiv 0$ , which is tantamount to neglecting energy storage and convection; then, this characteristic equation is simply given by

$$\left(\frac{dx}{dt}\right)_c = n\delta^2 = \frac{(\rho_l - \rho_v)g_x}{\mu_l} \delta^2 \quad (58)$$

The condensate thickness  $\delta$  is obtained from the further solution of equation (56); namely,

$$\frac{d\delta}{dx} = \frac{1}{f_2\langle\delta\rangle} \quad (59a)$$

and

$$\frac{d\delta}{dt} = \frac{1}{f_1\langle\delta\rangle} \quad (59b)$$

The solutions to equations (59a) and (59b), where the boundary conditions are given by

$$\begin{aligned} \delta\langle x, 0 \rangle &= 0 \\ \delta\langle 0, t \rangle &= 0 \end{aligned} \quad (60)$$

and it is assumed that  $S_3 \equiv 0$ , are given by equations (61):

$$\left. \begin{aligned} \delta &= \delta\langle x \rangle \\ x &= x\langle \delta \rangle = \frac{\rho_l(\rho_l - \rho_v)g_x L_{LV}}{\mu_l k_l (T_{LV} - T_c)} \left( \frac{\delta^4}{4} + \frac{bk_l \delta^3}{3} \right) \end{aligned} \right\} \quad (61a)$$

and

$$\left. \begin{aligned} \delta &= \delta\langle t \rangle \\ t &= t\langle \delta \rangle = \frac{\rho_l L_{LV}}{k_l (T_{LV} - T_c)} \left( bk_l \delta + \frac{\delta^2}{2} \right) \end{aligned} \right\} \quad (61b)$$

The single characteristic curve separates a steady-state solution region from a function-of-time-only solution region. The mathematical basis for this statement is that characteristic curves are curves of discontinuities in the derivatives of the dependent variables. Therefore, the characteristic equation (eq. (58)) determines where equations (61) apply for  $S \equiv 0$ . Equation (61a) describes the steady-state region, and equation (61b) describes the function-of-time-only region, where  $\delta$  is not a function of  $x$ . For a given value of time  $t$ , equation (61b) is solved for  $\delta$ , which is used in equation (58) to obtain the characteristic location. Equation (61a) is then solved for  $\delta$  for given values of  $x$  that are less than the characteristic location. The above calculation procedure gives  $\delta(x)$  at a given value of time  $t$ .

The assumed constancy of the parameters in equation (51) requires that the condensate thickness at the start of freezing  $\delta$  be constant. And the single characteristic solution for the two-phase region means that  $\delta$  will reach  $\delta_s$  uniformly with  $x$ , and freezing will initiate uniformly, for all values of  $x$  greater than the characteristic location. This characteristic location is where freezing initiates ( $X_s(t)$ ) in this instance. After the start of freezing, values of  $x \leq X_s(t)$  are in the steady-state region. Therefore, the point of freezing initiation  $X_s(t)$  does not move after the start of freezing (i. e.,  $X_s(t) = X_s = \text{Constant}$ ). This fixed location can be determined readily from a simple steady-state analysis of the condensing region.

A further consequence of these initial freezing conditions (i. e.,  $\delta(x > X_s, t_s) = \delta_s = \text{Constant}$ , and  $\Delta(x > X_s, t_s) = 0$ ) is determined by substituting them and equation (51) into equation (43). From this it is learned that  $\partial\Delta/\partial t(x > X_s, t_s) = 0$ . Axial conduction is a small effect, which is largest near  $X_s$  and tends to vanish at large  $x$ . If axial conduction had been included in the analysis, a  $\partial\Delta/\partial x$  term would have appeared in equation (43), and  $\partial\Delta/\partial t(x > X_s, t_s)$  would be greater than zero near  $X_s$ .

## Transient Condensing with Freezing Region

General problem. - The partial differential equations describing the three-phase region  $x > X_s(t)$  (eqs. (43) and (49)) can be rewritten in the following form by substituting equation (43) into equation (49) and neglecting the subcooling terms (i. e.,  $S_1 \equiv 0$  and  $S_2 \equiv 0$ ).

$$\frac{\partial\delta}{\partial t} + n\delta^2 \frac{\partial\delta}{\partial x} = \frac{A_6}{\delta} - \frac{A_5}{\Delta + A_2} \quad (62)$$

and

$$\frac{\partial \Delta}{\partial t} = A_4 \left( \frac{A_1}{\Delta + A_2} - \frac{A_3}{\delta} \right) \quad (63)$$

where

$$\left. \begin{aligned} A_1 &= k_s (T_{SL} - T_c) \\ A_2 &= k_s b \\ A_3 &= k_l (T_{LV} - T_{SL}) \\ A_4 &= \frac{1}{\rho_s L_{SL}} \\ A_5 &= \frac{A_1}{\rho_l L_{SL}} \\ A_6 &= \frac{A_3}{\rho_l} \left( \frac{1}{L_{LV}} + \frac{1}{L_{SL}} \right) \end{aligned} \right\} \quad (64)$$

Equations (62) and (63) are put in the following matrix form:

$$\frac{\partial \gamma}{\partial t} + B \frac{\partial \gamma}{\partial x} = C \quad (65a)$$

where

$$\frac{\partial \gamma}{\partial t} = \begin{bmatrix} \frac{\partial \Delta}{\partial t} \\ \frac{\partial \delta}{\partial t} \end{bmatrix} \quad (65b)$$

$$\frac{\partial \gamma}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial \Delta}{\partial \mathbf{x}} \\ \frac{\partial \delta}{\partial \mathbf{x}} \end{bmatrix} \quad (65c)$$

$$\mathbf{C} = \begin{bmatrix} A_4 \left( \frac{A_1}{\Delta + A_2} - \frac{A_3}{\delta} \right) \\ \frac{A_6}{\delta} - \frac{A_5}{\Delta + A_2} \end{bmatrix} \quad (65d)$$

and

$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & n\delta^2 \end{bmatrix} \quad (65e)$$

By satisfying the requirement that the determinant

$$|\mathbf{B} - \lambda \mathbf{I}| = 0 \quad (65f)$$

the following eigenvalues result:

$$\left. \begin{array}{l} \lambda_1 = 0 \\ \lambda_2 = n\delta^2 \end{array} \right\} \quad (66)$$

The differential equations are hyperbolic because these eigenvalues are real and unequal. The characteristic equations for the differential equations are

$$\left( \frac{d\mathbf{x}}{dt} \right)_I = 0 \quad (67a)$$

and

$$\left(\frac{dx}{dt}\right)_{\Pi} = n\delta^2 \quad (67b)$$

Because of the zero characteristic (eq. (67a)), which is a consequence of the assumption of no axial heat transfer, the commonly used characteristic method (e. g. , see refs. 9 and 10) cannot be used here. Fortunately, equations (62) and (63) are recognized to be in the normal form of references 11 and 12.

$$\frac{\partial\gamma}{\partial t} + \lambda \frac{\partial\gamma}{\partial x} = \text{Constant} \quad (68)$$

As a consequence of this normal form, only the  $\partial\gamma/\partial t$  derivative appears along the characteristic curves, and a solution can be obtained. Along the  $(dx/dt)_{\text{I}} = 0$  characteristic, the differential equation to solve is given by equation (69)

$$\frac{d\Delta}{dt} = A_4 \left( \frac{A_1}{\Delta + A_2} - \frac{A_3}{\delta} \right) \quad (69)$$

while along the  $(dx/dt)_{\text{II}} = n\delta^2$  characteristic,

$$\frac{d\delta}{dt} = \frac{A_6}{\delta} - \frac{A_5}{\Delta + A_2} \quad (70)$$

A solution to the problem now involves the numerical solution of equations (69) and (70), along the characteristics given by equations (67a) and (67b), respectively. The numerical approximations to equations (67), (69), and (70) are summarized in appendix B and discussed further in the section Direct Finite Difference Method.

Coupling term zero. - If the coupling term  $\partial\Delta/\partial t$  of equation (49) were zero, the partial differential equations would be decoupled. Equation (43) would then be simply an ordinary differential equation to solve at each position  $x$ , and equation (49) could be solved by the simple single characteristic method outlined previously for the condensing region. The coupling term is only exactly zero at the start of freezing and at steady state. Following the previous single characteristic procedure, the condensate layer for  $x > X_g$  is described by the following equations when the coupling term is neglected. The characteristic equation is given by

$$\left(\frac{dx}{dt}\right)_c = \frac{f_3(\delta)}{f_4(\delta)} = \frac{n\delta^2\left(\frac{3}{8}S_2 + 1\right)}{\frac{S_2}{2} + 1} \quad (71)$$

where  $\delta$  is obtained from the solution of equation (72) or (73).

$$\frac{d\delta}{dx} = \frac{1}{f_4(\delta)} = \frac{k_l(T_{LV} - T_{SL})}{\rho_l L_{LV} n \left(\frac{3}{8}S_2 + 1\right) \delta^3} \quad (72)$$

$$\frac{d\delta}{dt} = \frac{1}{f_3(\delta)} = \frac{k_l(T_{LV} - T_{SL})}{\rho_l L_{LV} \left(\frac{S_2}{2} + 1\right) \delta} \quad (73)$$

Assume that  $S_2 \equiv 0$ ; then the characteristic equation is given by

$$\left(\frac{dx}{dt}\right)_c = n\delta^2 \quad (74)$$

Equations (72) and (73) are readily integrated by using the boundary conditions

$$\begin{aligned} \delta(x, t_s) &= \delta_s = \text{Constant} \\ \delta(X_s, t) &= \delta_s = \text{Constant} \end{aligned} \quad (75)$$

The result of the integration, assuming that  $S_2 \equiv 0$ , is given by equations (76) and (77).

$$x - X_s = \frac{\rho_l L_{LV}}{k_l(T_{LV} - T_{SL})} \left( \frac{\delta^4 - \delta_s^4}{4} \right) \quad (76)$$

$$t - t_s = \frac{\rho_l L_{LV}}{k_l(T_{LV} - T_{SL})} \left( \frac{\delta^2 - \delta_s^2}{2} \right) \quad (77)$$

Therefore, by assuming the coupling term is zero, only algebraic equations need be

solved for  $\delta$  instead of partial differential equations. This would represent an appreciable computational savings. The equation for the solid layer thickness (eq. (43)) depends upon  $\delta$  at each  $x$ . Thus, it is solved as an ordinary differential equation at each  $x$ , where the condensate thickness is provided by equation (76) or (77).

## DIRECT FINITE DIFFERENCE METHOD

The solution of the partial differential equations (eqs. (43) and (49)) for the condensing-with-freezing region by a direct finite difference method is now discussed. The method could also be applied to the solution of the single partial differential equation (eq. (46)) of the condensing region should there be different boundary conditions.

A major problem with finite difference numerical methods is to select  $\Delta x$  and  $\Delta t$  increments such that the numerical solution is stable. This problem is considered in reference 13, where a method is given to calculate a time increment  $\Delta t$ , as a function of the  $\Delta x$  increment and the problem variables, which will result in a stable solution. Based on experience, a first-order finite difference approximation to the differential equations is employed. Reference 13 suggests that a backward difference be used for the spatial derivatives, with a forward difference for the time derivatives, and all coefficients evaluated at the previous time. The finite difference equations used to approximate equations (43) and (49) are found in appendix B. The following equation is obtained, by the method of reference 13, in order to calculate the time step required for a stable solution:

$$\Delta t = \frac{\Delta x}{2n\delta^2 N} \quad (78)$$

where  $N$  is an arbitrary number that is greater than 1. In order to give a constant value to  $\Delta t$ , so that layer profiles could be plotted at constant values of time, the terms of equation (78) are evaluated conservatively at the largest value of  $x$  for the problem (e. g.,  $x = L = 10$  ft or 3.05 m). This combination of finite difference approximations and time step resulted in a stable solution for the first time step and subsequent time steps. It was found that  $N = 1$  gave a very close estimate of the maximum allowable  $\Delta t$  that would permit a stable solution.

Stability does not assure accuracy. Small values of  $\Delta x$  and  $\Delta t$  (where  $\Delta t$  must always satisfy the stability criteria of eq. (78)) are chosen in order to reduce the truncation error without unduly increasing computing time (i. e., cost) and round-off error. The accuracy of the solution is best determined by comparing the results of the finite difference calculation to results of two limiting cases, where the calculation is inher-

ently accurate. These two cases, which have exact solutions, are a steady-state case and a case where the coupling term of equation (49) is assumed to be zero. By assuming that the coupling term is zero, the resulting single partial differential equation for  $\delta$  can then be solved by the simple single characteristic method with assured accuracy, since this is an exact solution. This analysis was worked out in the previous section. By also solving this same uncoupled differential equation directly by the finite difference method, an estimate of the finite difference method error, as compared to the equivalent exact solution, can be determined. In order to reduce computing time, it is desirable to use a small number of  $\Delta t$  and  $\Delta x$  increments. Figure 2 shows the percent error distribution along the plate, at two different times, for three different distributions of  $\Delta x$  elements, where a total of 500  $\Delta x$  increments were used for each distribution. This figure shows that the percent error is too large in the important region of the plate (i. e.,  $0.01 L < x < L$ ) for a uniform distribution of 500  $\Delta x$  increments in  $L = 10$  feet (3.05 m). The error in this region becomes acceptable if the 500  $\Delta x$  increments are heavily concentrated near  $X_s$  (i. e., many small increments are located near  $X_s$ ), where the major changes in the phase thickness profiles occur. Distribution B of figure 2 (150  $\Delta x$  in the first 0.05  $L$ , 100 in the next 0.1  $L$ , 50 in the following 0.1  $L$ , and 100 in the remainder of  $L$ ) was finally used in the numerical calculations of the coupled case; the results of which are described in appendix B. A time increment distribution, based on equation (78), was used with all of these  $\Delta x$  distributions (i. e.,  $N = 1000$  for the first 600  $\Delta t$  increments, then  $N = 100$  for the next 400  $\Delta t$ , and  $N = 10$  beyond that). This  $\Delta t$  distribution reduced the computation time (cost) while maintaining a stable and

TABLE I. - COMPARISON OF STEADY-STATE RESULTS OF REFERENCE 3 AND TRANSIENT

LARGE-TIME<sup>a</sup> RESULTS AT  $x = L = 10$  FEET (3.05 m)

Case	Thermal resistance of plate and coolant, $b$ , $(ft^2)(hr)(^{\circ}R)/Btu$ ; $(m^2)(K)/W$					
	$5 \times 10^{-4}$ ; $0.88 \times 10^{-4}$			$5 \times 10^{-3}$ ; $0.88 \times 10^{-3}$		
	Thickness of solidified layer, $\Delta$	Thickness of condensate layer, $\delta$	Position where freezing starts, $X_s$	Thickness of solidified layer, $\Delta$	Thickness of condensate layer, $\delta$	Position where freezing starts, $X_s$
	Percent error = $\left( \frac{ \text{This result} - \text{Reference 3 result} }{\text{Reference 3 result}} \right) \times 100$					
Uncoupled transient, $S \equiv 0$	0.125	0.1	3	0.1	0.1	0.6
Coupled transient, $S \equiv 0$	.6	.6	3	.5	.1	.6

<sup>a</sup>In the transient cases, steady state was considered to be when no change occurred in the third significant figures of  $\Delta$  and  $\delta$  at  $x = 10$  ft (3.05 m) for 500 calculations.

accurate solution ( $N \geq 1$ ). Table I contains a comparison of the approximate steady-state results from the finite difference calculation (i. e. ,  $\partial/\partial t \rightarrow 0$ ) and exact results from the closed-form steady-state analysis of reference 3. The agreement is good.

The characteristic solution of the coupled partial differential equations for the three-phase region was necessarily also solved by an approximate finite difference method. The accuracy and stability of this numerical solution are now compared to the previous direct finite difference solution results. One comparable case was computed, where the numerical approximations were of the same order and the parameters,  $\Delta x$  and  $\Delta t$  increments, and stability criteria were the same. The computed phase thicknesses were within 1/2 percent, and the two methods required very nearly the same computing time to reach the same problem time. It was found that equation (78), with  $N = 1$ , also gave a close estimate of the maximum allowable  $\Delta t$  for a stable solution for the coupled characteristic equations.

## RESULTS AND DISCUSSION

In this section the results of the transient analyses are discussed. The first part of this section is devoted to transient condensing of a pure vapor on a nonisothermal inclined plate where there is no freezing. The second part takes up transient condensing of a pure vapor with freezing.

### TRANSIENT CONDENSING OF A PURE VAPOR ON A NONISOTHERMAL INCLINED PLATE

The three-phase problem is made up of a two-phase region and a three-phase region. The problem of transient condensing on a nonisothermal inclined plate is the two-phase part of the whole solution. If the plate is too short to have freezing (i. e. ,  $X_s > L$ ) or  $T_c > T_{SL}$ , only the two-phase part of the whole problem need be considered. If it is further assumed that  $b = 0$ , then the special case of transient condensing on an isothermal plate results. This case was considered in reference 5. From the steady-state results of reference 3, which are discussed in appendix C, it is known that freezing is difficult to achieve during the condensation of a pure vapor on a vertical plate, no matter how cold the coolant. Generally, very low values of pressure or coolant thermal resistance, or a long plate inclined more toward the horizontal, or some small concentration of a noncondensable gas is necessary. Therefore, it can be expected that transient condensing with no freezing should be a more common occurrence for a vertical plate than condensing with freezing.

A single characteristic curve describes the two-phase condensing region. This curve, which is described by equation (57) (or, when  $S = 0$ , by eq. (58)), separates two solution regions in the  $x, t$  plane. In one region steady-state condensing occurs, and in the other region condensing is only a function of time. Figure 3 shows a group of such single characteristic curves for water condensed on a vertical plate at a given pressure and coolant temperature. The coolant and plate thermal resistance  $b$  is varied as a parameter. The characteristic curve indicates the time required to attain steady-state condensing at various positions  $x$  for a particular set of conditions. For the case considered in figure 3, a good representative time to attain steady-state for a 1-foot- (0.3-m-) long plate would be about 1 second. From equations (58) and (61b) or figure 3, it can be seen that it takes longer to attain steady state, at a given  $x$ , as the coolant and plate thermal resistance  $b$  increases or as the plate becomes more horizontal ( $g_x \rightarrow 0$ ). The effect of an increase in coolant temperature or pressure results in a much smaller increase in the time to attain steady state. The broken lines in figure 3 were determined from equations (57) and (59b) for the case where  $S_3 \neq 0$  in order to determine its effect. Clearly, the effect of subcooling could be neglected (i. e.,  $S_3 \equiv 0$ ).

## TRANSIENT CONDENSING WITH FREEZING ON

### A NONISOTHERMAL INCLINED PLATE

In this section, results of the less common but interesting situation of transient condensing with freezing are considered. This three-phase situation would tend to occur in practice when  $T_c < T_{SL}$ , and the pressure is low, the plate is nearly horizontal, or a small amount of noncondensable gas is present in the vapor. The additional complication caused by the inclusion of the noncondensable gas effect in a transient situation is beyond the scope of this report; however, it is analyzed for steady state in appendix C.

It was previously shown that a simple single characteristic solution would result only when the coupling term in equation (49) was neglected. This term, which accounts for the heat of vaporization liberated by the additional condensate mass needed to form the growing solid layer, is exactly zero only at the start of freezing and at steady state. During the transient process, the numerical value of the coupling term is about the same size as the  $\partial\delta/\partial t$  term in equation (49); therefore, it cannot generally be neglected in spite of the desirable simplification that results. The error caused by neglecting the coupling term is clearly shown by comparing the coupled (numerical solution) and uncoupled (single characteristic solution where there is no coupling term included) results of figure 4. Figures 4(a) and (b) display the growth rate of the condensate and solid layers using the coupled and uncoupled analyses at two positions on the plate ( $x = 5$  ft (1.5 m) and 10 ft (3m) for  $b = 5 \times 10^{-3}$  and  $5 \times 10^{-4}$  (ft<sup>2</sup>)(hr)(°R)/Btu ( $0.88 \times 10^{-3}$  and  $0.88 \times 10^{-4}$

$(m^2)(K)/W$ ), respectively). The coupling term, as shown in figure 4, retards the attainment of steady state. In the section, METHODS OF SOLUTION it was shown that  $\partial\Delta/\partial t \langle x > X_s, t_s \rangle = 0$  because axial conduction was neglected. The zero derivative is lost on figures 4(a) and (b) because of the scale of the plot. Figure 5(a) contains the phase thickness profiles ( $\delta\langle x \rangle$ ,  $\Delta\langle x \rangle$ ) at various times for the coupled solution where  $b = 5 \times 10^{-3} \text{ (ft}^2\text{)(hr)(}^\circ\text{R)/Btu}$  ( $0.88 \times 10^{-3} \text{ (m}^2\text{)(K)/W}$ ). Figure 5(b) contains a similar comparison for a lower value of  $b$ .

For a chronological description of the phase growth consider figure 5(a), which shows the phase growth history for a coupled solution where  $b = 5 \times 10^{-3} \text{ (ft}^2\text{)(hr)(R)/Btu}$  ( $0.88 \times 10^{-3} \text{ (m}^2\text{)(K)/W}$ ). Initially ( $t = 0$ ), the plate is cold but dry ( $\delta = 0$ ). The condensate layer starts to grow an instant later and with increasing time tends to "fill up" the steady-state profile until  $t = t_s$  when  $\delta$  reaches the condensate thickness required to start freezing  $\delta_s$ . From this point in time on, there exists a two-phase region above the fixed-start-of-freezing location at  $X_s$ , and a three-phase region below that point. The two-phase region, having reached steady state when freezing started, does not grow anymore. In the three-phase region, the condensate and solid layers continue to grow until they fill up their steady-state thickness profiles. Essentially, the same description applies to figure 5(b), where  $b = 5 \times 10^{-4} \text{ (ft}^2\text{)(hr)(}^\circ\text{R)/Btu}$  ( $0.88 \times 10^{-4} \text{ (m}^2\text{)(K)/W}$ ). However, freezing starts very near  $x = 0$  and the phases grow thicker.

Consider the special case of  $b = 0$ , which results in a constant wall temperature ( $T_w = T_c = \text{Constant}$ ). According to equation (51),  $\delta_s = 0$ , so that freezing starts immediately at  $t = 0$  and all along the plate. This academic case cannot be solved by the numerical methods discussed but can be very closely approximated by a case where  $b = 10^{-6} \text{ (ft}^2\text{)(hr)(}^\circ\text{R)/Btu}$  ( $0.18 \times 10^{-6} \text{ (m}^2\text{)(K)/W}$ ). The computing time is unfortunately very long for such a low value of  $b$ .

In spite of the inaccuracy of the uncoupled solution, this simple solution can give a crude estimate of the time to attain steady state. The error increases as  $b$  decreases. For example, according to figure 6 steady state is attained on a 1-foot (0.3-m) vertical plate in about 1 second.

## CONCLUDING REMARKS

Transient condensing on an initially dry inclined plate of no thermal capacity that is cooled by a fluid below the freezing point of the vapor was considered in this analysis. The problem was characterized by a two-phase condensing region and a three-phase region where the condensate freezes to form a growing solid layer. A Karman-Pohlhausen integral method, with linear temperature profiles across the phases was employed. This reduced the problem to the solution of simultaneous quasi-linear partial differential equations, where the condensate and solid thicknesses were the dependent

variables. A single characteristic curve, represented by algebraic equations, described the transient condensing (two phase) region. Families of characteristic curves, represented by ordinary differential equations, were involved in the solution for the three-phase region. A solution involving a finite difference equation approximation to the partial differential equations was also employed in the three-phase region for comparison. Small time and position increments were required for the numerical solution by either the finite difference or the characteristic method in the three-phase region in order to achieve an accurate and stable computation. This resulted in a long computing time. The stability, accuracy, and computation time of the characteristic and finite difference methods were found to be essentially equivalent. A single characteristic solution, which required no long computation, was possible in the three-phase region only when the term coupling the two describing partial differential equations was neglected. This coupling term accounts for the heat liberated by the extra liquid that must be condensed in order to form the growing solid layer. An appreciable error can occur when this coupling term is neglected. This term is negligible only at the start of freezing and at steady state.

The results of this analysis indicated that freezing starts at and beyond a particular point on the cold plate as soon as the insulating condensate layer is thick enough for the wall temperature to drop to the freezing temperature at that point. The location of the start of freezing does not change after freezing begins. This fixed location is a consequence of the single characteristic solution for the transient condensing region. The solid and condensate layers, which have a shape similar to boundary layers, were found to grow until they attain a steady-state profile.

The analysis of transient condensing with freezing was shown to specialize to a number of interesting cases which are discussed in appendix C. Some examples are transient condensing and freezing on a horizontal plate, transient condensing on a non-isothermal inclined plate, and steady-state condensing and freezing (or melting) on a nonisothermal inclined plate. With minor changes in the initial conditions, the analysis could handle transient condensing with melting; and with a minor modification, transient freezing of a liquid flowing past a cold plate could be adequately analyzed.

Lewis Research Center,  
National Aeronautics and Space Administration,  
Cleveland, Ohio, March 26, 1969,  
120-27-04-27-22.

## APPENDIX A

### SYMBOLS

$A_1, \dots, A_6$	constants, defined by eq. (64)
$b$	thermal resistance of plate and coolant, $b = 1/h_c + d_m/k_m$ , (ft <sup>2</sup> )(hr)(°R)/Btu; (m <sup>2</sup> )(K)/W
$c$	specific heat at constant pressure, Btu/(lbm)(°F); J/(kg)(K)
$d_m$	thickness of plate, ft; m
$f_1\langle\delta\rangle, f_2\langle\delta\rangle$	functions of $\delta$
$g_c$	standard conversion factor, ft/hr <sup>2</sup> ; m/sec <sup>2</sup>
$g_o$	steady acceleration of gravity, ft/hr <sup>2</sup> ; m/sec <sup>2</sup>
$g_x$	steady body force acceleration along inclined plate, $g_x = g_o \sin \varphi$ , ft/hr <sup>2</sup> ; m/sec <sup>2</sup>
$h$	surface heat-transfer coefficient, Btu/(ft <sup>2</sup> )(hr)(°R); W/(m <sup>2</sup> )(K)
$I$	unit diagonal matrix
$k$	thermal conductivity, Btu/(ft)(hr)(°R); W/(m)(K)
$L$	length of plate, ft; m
$L_{LV}, L_{LV}\langle P \rangle$	heat of vaporization, Btu/lbm; J/kg
$L_{SL}$	heat of fusion, Btu/lbm; J/kg
$(m/A)_{cd}, (m/A)_{fr}$	mass flux (condensing and freezing), lbm/(hr)(ft <sup>2</sup> ); kg/(m <sup>2</sup> )(sec)
$(m/A)_{LV}$	mass flux across liquid-vapor interface in y-direction, lbm/(hr)(ft <sup>2</sup> ); kg/(m <sup>2</sup> )(sec)
$(m/A)_{SL}$	mass flux across liquid-solid interface in y-direction, lbm/(hr)(ft <sup>2</sup> ); kg/(m <sup>2</sup> )(sec)
$N$	number greater than 1
$n$	$[(\rho_l - \rho_v)g_x]/\mu_l$ , 1/(ft)(hr); 1/(m)(sec)
$P$	pressure, psia; N/m <sup>2</sup>
$(Q/A)_c$	heat flux to coolant, Btu/(ft <sup>2</sup> )(hr); W/m <sup>2</sup>
$q$	heat flux, positive in y-direction, Btu/(ft <sup>2</sup> )(hr); W/m <sup>2</sup>
$S_1, S_2, S_3$	subcooling parameters, defined by eqs. (42), (48), and (45)

$T$	temperature, $^{\circ}\text{R}$ ; K
$T_i$	interface temperature, $^{\circ}\text{R}$ ; K
$T_{LV}$	boiling temperature, $^{\circ}\text{R}$ ; K
$T_{SL}$	freezing temperature, $^{\circ}\text{R}$ ; K
$T_v$	superheated vapor temperature far from plate, $^{\circ}\text{R}$ ; K
$T_w, T_w(x, t)$	wall temperature (temperature of the surface of plate in contact with phases), $^{\circ}\text{R}$ ; K
$t$	time from start of condensation, hr; sec
$\Delta t_j$	nonconstant time increment, defined in appendix B, hr; sec
$t_s$	time when freezing starts, hr; sec
$U_v$	bulk vapor velocity, ft/hr; m/sec
$u_l$	velocity of liquid in x-direction relative to fixed axes, ft/hr; m/sec
$v_l$	velocity of liquid in y-direction, relative to fixed axes, ft/hr; m/sec
$W$	noncondensable gas mass fraction
$X_s, X_s(t)$	position along plate where freezing starts, ft; m
$x$	x-coordinate direction along plate from start of cold section, ft; m
$x_A, x_R, x_C$	points in mesh, defined in appendix B
$\Delta x_i$	nonconstant $x$ increment, defined in appendix B, ft; m
$(dx/dt)_I, (dx/dt)_c$	slope of characteristic curves
$Y_{SL}, Y_{SL}(x, t)$	location of melt surface (SL) from plate, ft; m
$Y_{LV}, Y_{LV}(x, t)$	location of condensing surface (LV), ft; m
$y$	y-coordinate, normal to plate, ft; m
$\gamma$	term defined by eq. (65)
$\Delta, \Delta(x), \Delta(x, t)$	thickness of solidified layer, ft; m
$\delta, \delta(x), \delta(x, t)$	thickness of condensate layer, ft; m
$\delta_s$	thickness of condensate layer where freezing starts, ft; m
$\mu$	dynamic viscosity, lbm/(ft)(hr); N-sec/m <sup>2</sup>
$\rho$	mass density, lbm/ft <sup>3</sup> ; kg/m <sup>3</sup>
$\varphi$	angle of plate from horizontal, rad

**Subscripts:**

<b>c</b>	<b>coolant</b>
<b>cd</b>	<b>condensing</b>
<b>fr</b>	<b>freezing</b>
<b>i</b>	<b>indicates x-direction in numerical calculations</b>
<b>j</b>	<b>indicates t-direction in numerical calculations</b>
<b>l</b>	<b>liquid</b>
<b>m</b>	<b>plate</b>
<b>s</b>	<b>solid</b>
<b>v</b>	<b>vapor</b>

## APPENDIX B

### DISCUSSION OF NUMERICAL CALCULATIONS

The numerical calculations for the finite difference and characteristic solutions of the three-phase region are discussed further in this appendix. In particular, the finite difference approximations to the coupled partial differential equations in the text are described.

#### CHARACTERISTIC CALCULATION

The characteristic solution of the three-phase region involves the solution of equation (69) along characteristic I, given by equation (67a), while equation (70) is solved along characteristic II, given by equation (67b). Consider figure 7, which shows typical characteristic curves for this problem and the grid used for the calculation. The values of  $\delta_{i,j}$  ( $\delta$  at point B),  $\delta$  at point R, and  $x_R$  are to be determined. To solve equation (70) along the II characteristic, given by equation (67b), linear approximations to these equations between points B and R are used. The value of  $x$  at point R is determined from the following approximation to equation (67b):

$$x_R = x_C - n(\delta_{i,j-1})^2 \Delta t_j \quad (79)$$

while  $\delta_{i,j}$  is determined by the following approximation to equation (70)

$$\delta_{i,j} = \delta_R + \left( \frac{A_6}{\delta_R} - \frac{A_5}{\Delta_R + A_2} \right) \Delta t_j \quad (80)$$

Values of  $\delta_R$  and  $\Delta_R$  are determined by the following linear interpolations at  $x_R$  between values of  $\delta$  and  $\Delta$  at  $x_A$  and  $x_C$

$$\delta_R = (\delta_{i,j-1}) \left[ 1 - n \frac{\Delta t_j}{\Delta x_i} (\delta_{i,j-1})^2 \right] + n \frac{\Delta t_j}{\Delta x_i} (\delta_{i,j-1})^2 (\delta_{i-1,j-1}) \quad (81a)$$

and

$$\Delta_R = (\Delta_{i,j-1}) \left[ 1 - n \frac{\Delta t_j}{\Delta x_i} (\delta_{i,j-1})^2 \right] + n \frac{\Delta t_j}{\Delta x_i} (\delta_{i,j-1})^2 (\Delta_{i-1,j-1}) \quad (81b)$$

Since the intervals are not constant,  $\Delta t_j = t_j - t_{j-1}$  and  $\Delta x_i = x_i - x_{i-1}$ . Now the value of  $\Delta$  at point B  $\Delta_{i,j}$  is determined by solving equation (69) along the I characteristic. This solution simply involves the solution of a quadratic equation in  $\Delta_{i,j}$ , for the largest positive root, making use of the previously calculated values of  $\delta_{i,j}$ .

$$(\Delta_{i,j})^2 + \Delta_{i,j} \left( A_2 + \frac{\Delta t_j A_3 A_4}{\delta_{i,j}} - \Delta_{i,j-1} \right) - \left[ A_2 \Delta_{i,j-1} + \Delta t_j A_4 \left( A_1 - \frac{A_2 A_3}{\delta_{i,j}} \right) \right] = 0 \quad (82)$$

## FINITE DIFFERENCE CALCULATION

The direct finite difference solution to equations (62) and (63) involves an explicit solution where a first-order forward difference is used for the time derivatives and a first-order backward difference is used for the  $x$  derivatives (see fig. 8).

$$\left. \begin{aligned} \frac{\partial \delta}{\partial t} &\cong \frac{\delta_{i,j+1} - \delta_{i,j}}{\Delta t_j} \\ \frac{\partial \Delta}{\partial t} &\cong \frac{\Delta_{i,j+1} - \Delta_{i,j}}{\Delta t_j} \\ \frac{\partial \delta}{\partial x} &\cong \frac{\delta_{i,j} - \delta_{i-1,j}}{\Delta x_i} \end{aligned} \right\} \quad (83)$$

The coefficients in the partial differential equations that involve the dependent variables are evaluated as  $\delta_{i,j}$  and  $\Delta_{i,j}$ . Equation (62) is approximated by the following difference equation which is solved explicitly for the unknown  $\delta_{i,j+1}$ .

$$\delta_{i,j+1} = \delta_{i,j} + \Delta t_j \left[ \frac{A_6}{\delta_{i,j}} - \frac{A_5}{\Delta_{i,j} + A_2} - n (\delta_{i,j})^2 \left( \frac{\delta_{i,j} - \delta_{i-1,j}}{\Delta x_i} \right) \right] \quad (84)$$

Equation (63) is similarly approximated, resulting in a quadratic equation for  $\Delta_{i,j+1}$ , where  $\delta_{i,j+1}$  is supplied from equation (84) and the largest positive root is used.

$$0 = (\Delta_{i,j+1})^2 + \Delta_{i,j+1} \left( A_2 + \frac{\Delta t_j A_3 A_4}{\delta_{i,j+1}} - \Delta_{i,j} \right) - \left[ A_2 \Delta_{i,j} + \Delta t_j A_4 \left( A_1 - \frac{A_2 A_3}{\delta_{i,j+1}} \right) \right] \quad (85)$$

This numerical calculation procedure will not start up at the start of freezing (i. e. , at  $t = t_g$ ). Fortunately, the coupling term is zero at this time so that the single characteristic approximation can be used in calculating the first time step. Once the first step is calculated, the numerical procedure is used.

## APPENDIX C

### DISCUSSION OF SPECIAL CASES

An interesting feature of the three-phase problem is the wealth of special cases that can result by simplifications of, or minor modifications to, the partial differential equations. Some of the special cases have not appeared in the literature.

Two of the special cases have been discussed in the main text; they are the three-phase transient problem, where the inclined plate is isothermal, and transient condensing on a nonisothermal inclined plate. Two special cases that have been analyzed in reference 3 are summarized here because the results give the reader a more complete understanding of the physics of the problem. These cases are transient condensing and freezing on a horizontal plate and steady-state condensing and freezing (or melting) of a pure vapor on an inclined plate. The important effect of a noncondensable gas on the steady-state problem was also considered in reference 3; it too is summarized in this section.

Three other transient special cases are discussed that require minor modifications to the describing partial differential equations: melting of a thin slab of solid by its vapor, freezing of a liquid that flows past a cold plate, and finally the three-phase problem when the coolant and vapor parameters (e.g.,  $T_c$ ,  $h_c$ , and  $P$ ) are known functions of time and/or position. No numerical results are worked out for these three cases. A major modification is required for a case where there is surface evaporation, or where the plate has finite thermal capacity.

### STEADY-STATE CONDENSING WITH FREEZING (MELTING) ON A NONISOTHERMAL PLATE

In this section, the special case of steady state is considered, and the effect of parameters upon condensing with freezing is demonstrated in detail. This special case has been considered in reference 3 and is summarized herein in order to give a more complete physical discussion of the three-phase problem. By neglecting the time derivative terms and energy convection (i.e.,  $S_2 \equiv 0$ ,  $S_3 \equiv 0$ ), equations (46), (43), and (49) directly simplify to the steady-state equations of reference 3, which describe steady-state condensing of a pure vapor on a nonisothermal inclined plate. These equations are manipulated and solved to show the conditions for freezing. The effect of noncondensable gas on freezing is also demonstrated by a minor modification to the steady-state analysis.

## Pure Vapor

Setting the time derivative terms equal to zero, neglecting energy convection (i. e.,  $S_2 \equiv 0$ ,  $S_3 \equiv 0$ ), and integrating according to the boundary conditions (eqs. (52) and (53)) results in the following steady-state equations for the condensate liquid and solid layer profiles. Where there is no solid layer, at  $x \leq X_s$ ,

$$\frac{\delta^4}{4k_l} + \frac{b\delta^3}{3} = \frac{\mu_l(T_{LV} - T_c)x}{\rho_l L_{LV}(\rho_l - \rho_v)g_x} \quad (86)$$

Where there is a solid layer, at  $x > X_s$ , the following are the condensate and solid layer thickness profile equations:

$$\frac{\delta^4 - \delta_s^4}{4k_l} = \frac{\mu_l(T_{LV} - T_{SL})(x - X_s)}{\rho_l L_{LV}(\rho_l - \rho_v)g_x} \quad (87)$$

and

$$\Delta = k_s \left[ \frac{\delta}{k_l} \left( \frac{T_{SL} - T_c}{T_{LV} - T_{SL}} \right) - b \right] \quad (88)$$

where  $\delta_s$  is given by equation (51) and the wall temperatures by equations (32) and (33).

According to equation (51), these results simplify further for a case where the condensate is warmer than the freezing point (i. e.,  $T_c \geq T_{SL}$ ), since no freezing is possible ( $\delta_s \leq 0$ ). In this case, the remaining equation (eq. (86)) describes steady-state condensing on a nonisothermal inclined plate. If it is further assumed that  $b \equiv 0$ , so that the wall temperature is constant, the equation is exactly the one solved by Nusselt (ref. 6).

Equations (32), (33), (51), and (86) to (88) were solved for  $T_w$ ,  $\delta_s$ ,  $\delta$ , and  $\Delta$  to indicate the effect of the system parameters for a water system. Figure 8(a) contains the nominal case from which one parameter is varied for each subsequent part of figure 8. From figures 8(a) to (d), it is clear that an increase in coolant temperature, pressure, and/or coolant thermal resistance  $b$  reduces the condensate thickness by a small amount. However, this small change can greatly reduce the solid thickness and the possibility of having a solid form on a given plate.

The heat flux in all cases of figure 8 is of the order of  $10^5$  Btu/(ft<sup>2</sup>)(hr) ( $3.15 \times 10^5$  W/m<sup>2</sup>), which is above the "burnout flux" for many cryogenic coolants (e. g., see ref. 14). Therefore, the cases of figure 8 would be limited to a value of  $b$  correspond-

ing to film boiling (e. g. ,  $b < 0.01 \text{ (ft}^2\text{)(hr)}(^{\circ}\text{R})/\text{Btu}$  or  $1.8 \times 10^{-3} \text{ (m}^2\text{)(K)}/\text{W}$ ) in many real cases. For an example, consider a liquid-nitrogen-cooled vertical plate at the conditions of figure 8(d); clearly, there is no freezing on the plate. A considerable reduction in pressure, or a reorientation of the plate toward the horizontal would again result in freezing on the plate for the conditions of figure 8(d). Therefore, in summary, the coolant thermal resistance  $b$  would not, generally, be small enough in practice to allow a solid layer to form on a short vertical plate from the pure slowly moving vapor, no matter how cold the coolant, unless the pressure is very low or the plate is nearly horizontal.

### Effect of a Small Quantity of Noncondensable Gas

If a small quantity of noncondensable gas (e. g. , air) is present in the vapor (e. g. , steam), the air would concentrate at the liquid-vapor interface because of the slow diffusion of air back into the bulk vapor. The air concentration would lower the partial pressure of the water vapor ( $P_i < P$ ) at the liquid-vapor interface, thereby lowering the liquid-vapor interface temperature  $T_i(P_i)$  below saturation  $T_{LV}$ . This decreases the driving temperature difference across the condensate layer  $T_i - T_w$  and effectively increases the insulating ability of the condensate layer. Reference 15 has analyzed the effect of a noncondensable gas (air) on condensing of a vapor (steam) upon a constant temperature vertical cold plate ( $T_w' = \text{Constant}$ ). The results of the analysis for air in steam can be adapted to this analysis to determine  $T_i$ . The wall of reference 15 is essentially replaced here by the solidified layer where  $T_w' \equiv T_{SL} = \text{constant}$ . With  $T_i$  known, as a function of the total pressure  $P$  and the air mass fraction  $W$ , the effect of  $W$  upon  $\delta_s$ ,  $X_s$ ,  $\delta$ , and  $\Delta$  can be determined by replacing  $T_{LV}$  by  $T_i$  in equations (51) and (86) to (88), respectively. By using the curves from reference 15, values of  $T_i$  are determined for  $W = 0$  (pure vapor), 0.01, and 0.05 and  $P = 1.7$  and 14.7 psia

TABLE II. - INTERFACE TEMPERATURE  
CALCULATED FROM REFERENCE 15

Total pressure, P		Noncondensable gas mass fraction, W					
		0 (T <sub>i</sub> = T <sub>LV</sub> )		0.01		0.05	
		Interface temperature, T <sub>i</sub>					
psia	N/m <sup>2</sup>	°F	K	°F	K	°F	K
14.7	10 <sup>5</sup>	212	373	92.5	307	65.5	292
1.7	1.17×10 <sup>4</sup>	120	322	44.0	280	38.5	278

( $11.7 \times 10^3$  and  $10^5$  N/m<sup>2</sup>). These values are listed in table II. With these values of  $T_i$ , the phase thicknesses shown in figure 9 are obtained. Clearly, a small fraction of non-condensable gas greatly increases the solid thickness and the chance of having a solid layer at a given position  $x'$ .

## TRANSIENT CONDENSING WITH FREEZING ON A HORIZONTAL COLD PLATE

Far out along the inclined plate, or for a horizontal plate, where the  $\partial/\partial x$  terms are negligible, the problem becomes a function of time only. The equations describing this case are obtained by setting  $\partial/\partial x \equiv 0$  in the governing partial differential equations (eqs. (43), (46), and (49)). These resulting equations are the same as those found in reference 3, where the Karman-Pohlhausen method was used. Figure 10 contains a plot from reference 3 of the phase growth rates for various conditions. Figure 10 shows that for  $b > 0$  a condensate layer grows until it is thick enough for the wall temperature to reach the freezing point, at which time a solid layer starts to grow. As  $b$  approaches zero, this time lag goes to zero such that at  $b = 0$  the solid and condensate both start

TABLE III. - ERROR IN CALCULATION OF CONDENSING SURFACE LOCATION BY  
VARIOUS METHODS FOR ONE-DIMENSIONAL CASE

WHERE  $T_w = \text{CONSTANT}$  (OR  $b = 0$ )

Time, min	Exact solution for superheated vapor <sup>a</sup>	Karman-Pohlhausen method with linear temperature profile, saturated vapor, and nearly zero thermal resistance, <sup>b</sup>	
		$S \neq 0$	All $S \equiv 0$
		Percent error in $Y_{LV}$ compared to exact solution for a saturated vapor	
0.06	2.5	1.5	9
.3	↓	1.3	8
.6		↓	
3.0			
6.0		↓	
30.0		1.2	
60.0		1.2	↓

<sup>a</sup>Superheat = 125° F (70 K).

<sup>b</sup>Thermal resistance,  $b = 10^{-6}(\text{ft}^2)(\text{hr})(^\circ\text{R})/\text{Btu}$  or  $0.18 \times 10^{-6}(\text{m}^2)(\text{K})/\text{W}$  is an adequate approximation to  $b \equiv 0$ , and is computationally necessary. The wall temperature  $T_w$  is constant for  $b \equiv 0$ . This limiting case is equivalent to the exact solution where  $T_w \equiv \text{Constant}$ .

growing at the same time. The growth rates of both phases, for  $b = 0$  and for large time at any value of  $b$ , are proportional to  $(\text{time})^{1/2}$ . The results of these one-dimensional equations were compared in table III (taken from ref. 3) for the special case of  $b \cong 0$  (i.e., a nearly constant wall temperature) to the corresponding constant wall temperature exact solution. This comparison shows that the approximate Karman-Pohlhausen integral method with  $S \neq 0$  gave results that are very close to the exact solution results. The comparison in table III also indicates that even a greatly superheated vapor has little effect.

## NEGLECTIBLE SUBCOOLING

By neglecting the energy storage and thermal energy convection terms in the energy equation in comparison to the conduction terms, a considerable analytical simplification is possible. By this assumption, equation (3), for example, reduces to

$$\frac{\partial^2 T_l}{\partial y^2} = 0 \quad (89)$$

which results in a linear temperature profile. This type of assumption is used for each phase layer and is generally restricted to cases where the phase layers are thin and there is a phase change. Carrying out the analysis as before, with equation (89) replacing equation (3), results in differential equations for the condensate layer growth that are much more easily derived. The same equations can also be derived from equations (43), (46), and (49) by setting  $S_1$ ,  $S_2$ , and  $S_3$  equal to zero.

Neglecting all the subcooling terms ( $S \cong 0$ ) can produce a significant error, as shown by table III; however, if the subcooling in the solid layer ( $S_1$ ) is not neglected, this error becomes acceptable.

## MELTING OF A THIN SLAB OF SOLID BY ITS VAPOR

Consider a thin slab of solid (e.g., ice) of initial thickness  $\Delta(x, 0)$  that is attached to an inclined plate where the coolant is at constant  $T_c$  and  $b$ . Saturated steam is suddenly introduced, and the ice melts while the vapor condenses and the liquid runs down over the plate and ice layer. The ice melts most rapidly at the leading edge where the insulating condensate layer is the thinnest. In time, steady state will be attained, where there will be a condensate layer and perhaps a solidified layer over some portion

of the plate. For the present analysis to fit this physical case, the initial solid layer must be thin so that the time lag required to heat the solid to a linear temperature profile is small compared to the time required to reach steady state. The previous analysis (eqs. (43), (46), and (49)) can inherently handle melting as well as freezing. Whether there is melting or freezing depends upon the boundary and initial conditions compared to steady state. Equation (43) indicates that melting (i. e.,  $\partial\Delta/\partial t$  is negative) will occur if the initial ice thickness is greater than that of steady state for those same conditions. If the initial thickness is smaller than that demanded by steady state, freezing will occur. The same differential equations and numerical program will therefore handle changes from one steady state to another, requiring only minor modifications, such as in the initial conditions.

## TRANSIENT FREEZING OF A FLOWING LIQUID UPON A COLD PLATE

Consider a liquid flowing over a cold flat plate where conditions suddenly change somewhat such that a solid layer begins to form and grow on the plate from the liquid. Equation (43) for the solid layer, which was derived on the assumption that  $T_s(y)$  is nearly linear, is adapted to this problem by replacing the condensate layer thermal resistance  $\delta/k$  by an equivalent resistance for the flowing liquid  $1/h_l$ ; and  $T_{LV}$  is replaced by  $T_l$  as indicated by equation (90).

$$\rho_s L_{SL} \left\{ 1 - \frac{S_1}{2} \left[ \left( \frac{k_s b}{k_s b + \Delta} \right)^2 - 1 \right] \right\} \frac{d\Delta}{dt} = -h_l (T_l - T_{SL}) + \frac{T_{SL} - T_c}{\frac{\Delta}{k_s} + b} \quad (90)$$

There are considerable difficulties in estimating  $h_l$  for this transient turbulent liquid flow, where mass is removed from the boundary layer (i. e., suction) to form the growing solid layer. It is therefore probable that the approximations concerned with the solid layer that were needed to obtain equation (43) are of relatively minor significance. The simplest assumption is that  $h_l$  is time-invariant. Indeed, the experimental study of reference 1 indicated that  $h_l$  was approximately time-invariant, at least at large  $x$ . Theoretical or empirical results for flow past a plate could then be used to estimate  $h_l$ . Since  $h_l$  is a function of position, decreasing with  $x$ , the solid layer will have the shape of a boundary layer. The solution of this problem involves solving the above ordinary differential equation (eq. (90)) in time at a number of  $x$  positions, beyond where freezing starts at  $X_s$ . The parameters  $T_l$ ,  $T_c$ , and  $b$  are described by known constants, differential equations, or functions. The start of freezing  $X_s$  is determined from the location where the wall temperature reaches the freezing point.


$$T_w \langle x=X_s \rangle = T_{SL} = T_l - \frac{T_{SL} - T_c}{bh_l} \quad (91)$$

## NONCONSTANT PARAMETERS

In the formation of the three-phase analytical model, it was assumed that the coolant thermal resistance  $b$ , coolant temperature  $T_c$ , and system pressure  $P$  were held constant during the transient process. It was also assumed that the plate had negligible thermal capacity. These assumptions resulted in a single direction of mass flow (e. g., freezing, but not freezing followed by melting). By relaxing the constant parameter restrictions such that the parameters  $P$ ,  $T_c$ , and  $b$  are known functions of time and position, it is possible to have a nonconstant  $X_s$ , and also freezing at one time and place and melting elsewhere. There is no inherent reason why the numerical program for the differential equations could not be simply modified to handle this case of known functions of  $P \langle x, t \rangle$ ,  $T_c \langle x, t \rangle$ , and  $b \langle x, t \rangle$ . Accounting for the thermal storage of the plate, even where it is assumed that the plate is a simple thermal capacitor of no thermal resistance, would require major changes in the present analysis.

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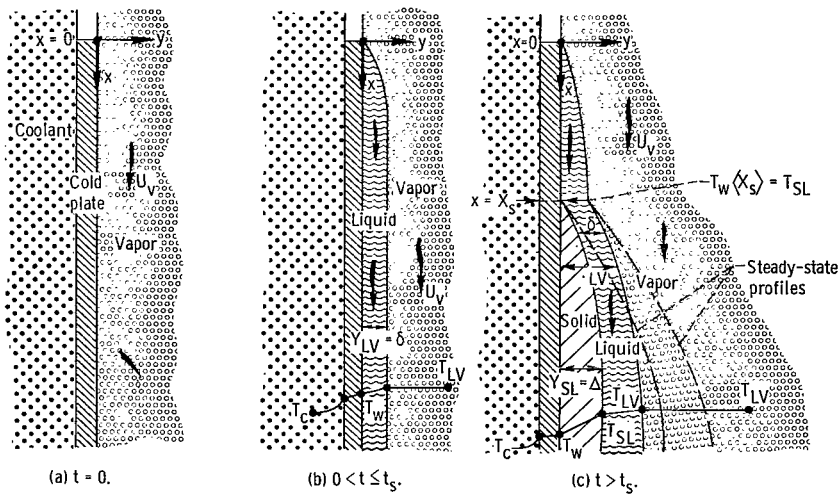


Figure 1. - Stages of phase layer growth on vertical cold plate where coolant temperature is below freezing temperature ( $T_c < T_{SL}$ ).

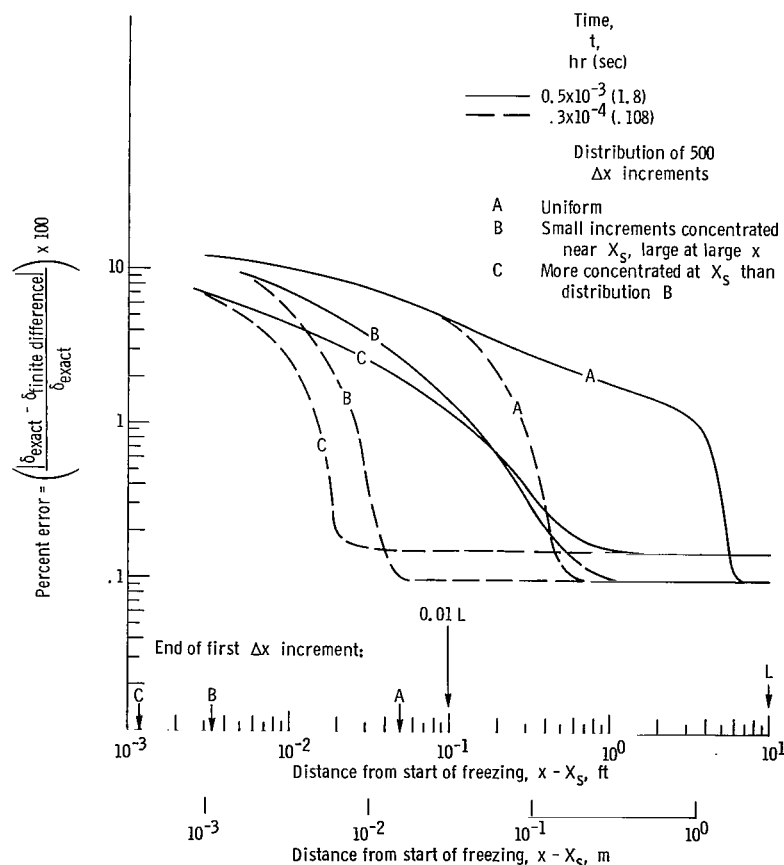


Figure 2. - Changes in percent error distribution caused by changing concentration of total of 500 small and large  $\Delta x$  increments over 10-foot- (3.05-m-) long plate. Position where freezing starts,  $X_s = 3.82 \times 10^{-3}$  feet (1.16  $\times 10^{-3}$  m).

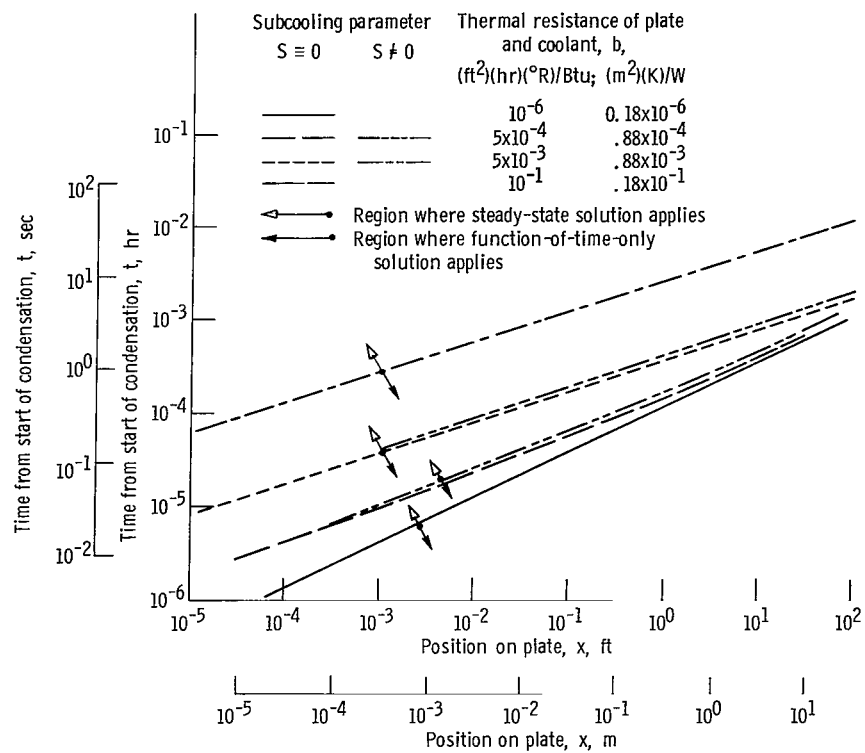
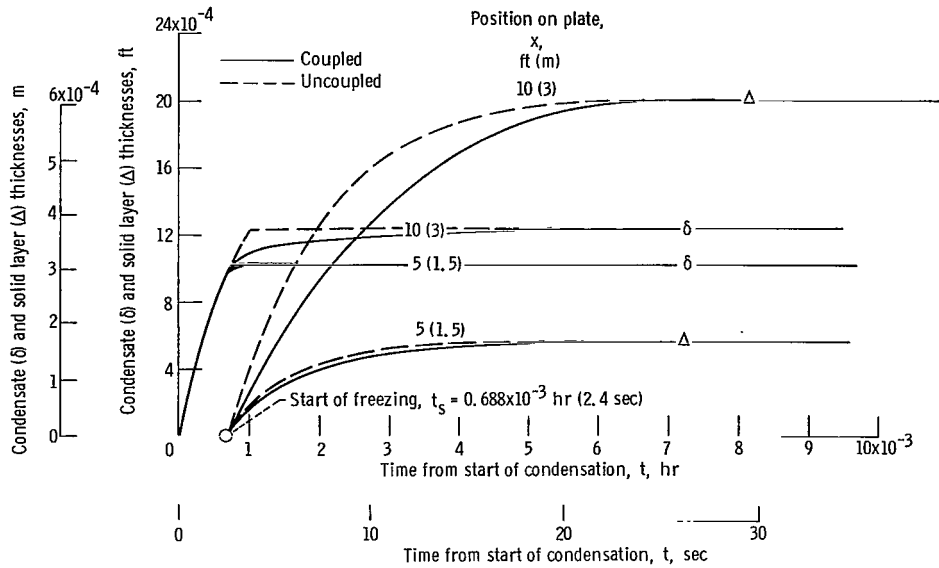
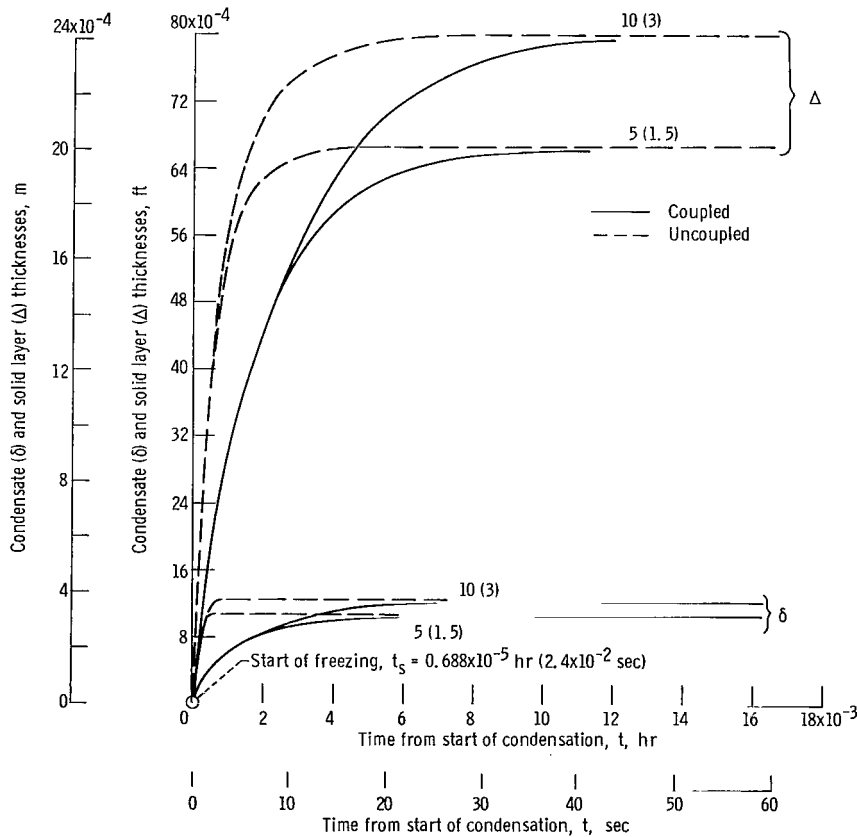


Figure 3. - Single characteristic curves for transient condensing of water on a nonisothermal vertical plate. Pressure, 14.7 psia ( $10^5 \text{ N/m}^2$ ); coolant temperature,  $140^\circ \text{R}$  ( $77 \text{ K}$ ).

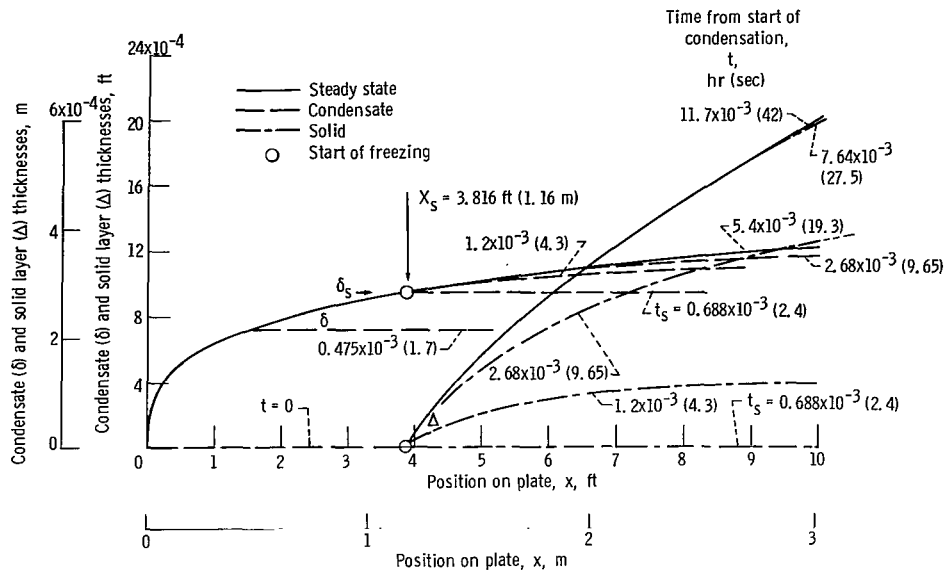


(a) Thermal resistance of plate and coolant,  $5 \times 10^{-3}$  (ft<sup>2</sup>)(hr)(°R)/Btu ( $0.88 \times 10^{-3}$  (m<sup>2</sup>)(K)/W).

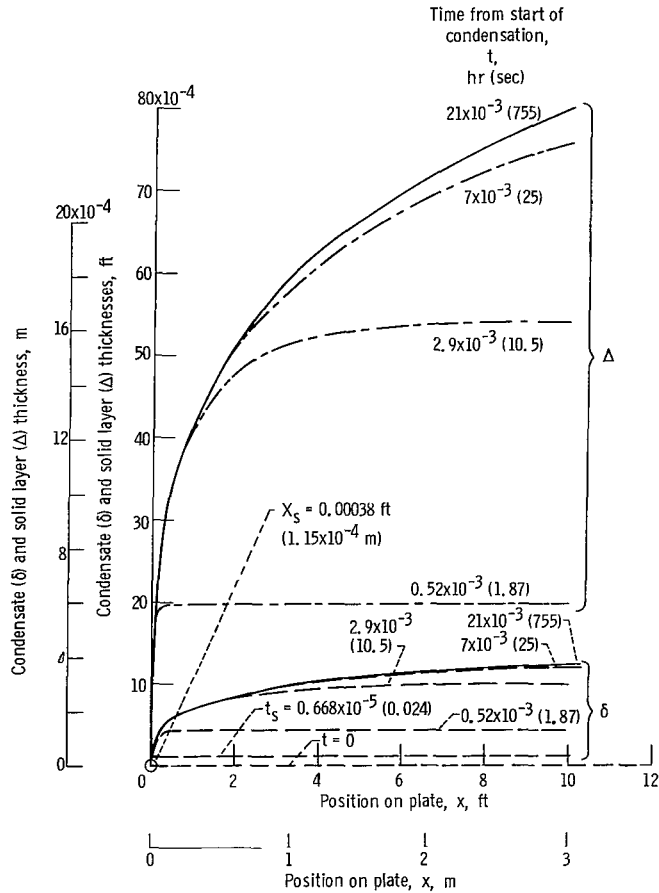


(b) Same as 4(a) but thermal resistance of plate and coolant,  $5 \times 10^{-4}$  (ft<sup>2</sup>)(hr)(°R)/Btu ( $0.88 \times 10^{-4}$  (m<sup>2</sup>)(K)/W).

Figure 4. - Comparison of phase growth histories at 5 feet (1.53 m) and 10 feet (3.05 m) along plate, by coupled and uncoupled computations. Water; all  $S \equiv 0$ ; pressure, 14.7 psia ( $10^5$  N/m<sup>2</sup>); coolant temperature, 140° R (77 K).



(a) Thermal resistance,  $b = 5 \times 10^{-3} \text{ (ft}^2\text{)(hr)(}^\circ\text{R)/Btu}$  ( $0.88 \times 10^{-3} \text{ (m}^2\text{)(K)/W}$ ).



(b) Same as 5(a) but thermal resistance,  $b = 5 \times 10^{-4} \text{ (ft}^2\text{)(hr)(}^\circ\text{R)/Btu}$  ( $0.88 \times 10^{-4} \text{ (m}^2\text{)(K)/W}$ ).

Figure 5. - Coupled calculation of condensate and solid layer thickness profiles at various times after start of condensing of vapor upon initially dry vertical cold plate. All subcooling parameters  $S = 0$ ; pressure,  $14.7 \text{ psia (10}^5 \text{ N/m}^2)$ ; coolant temperature,  $140^\circ \text{ R (77 K)}$ .

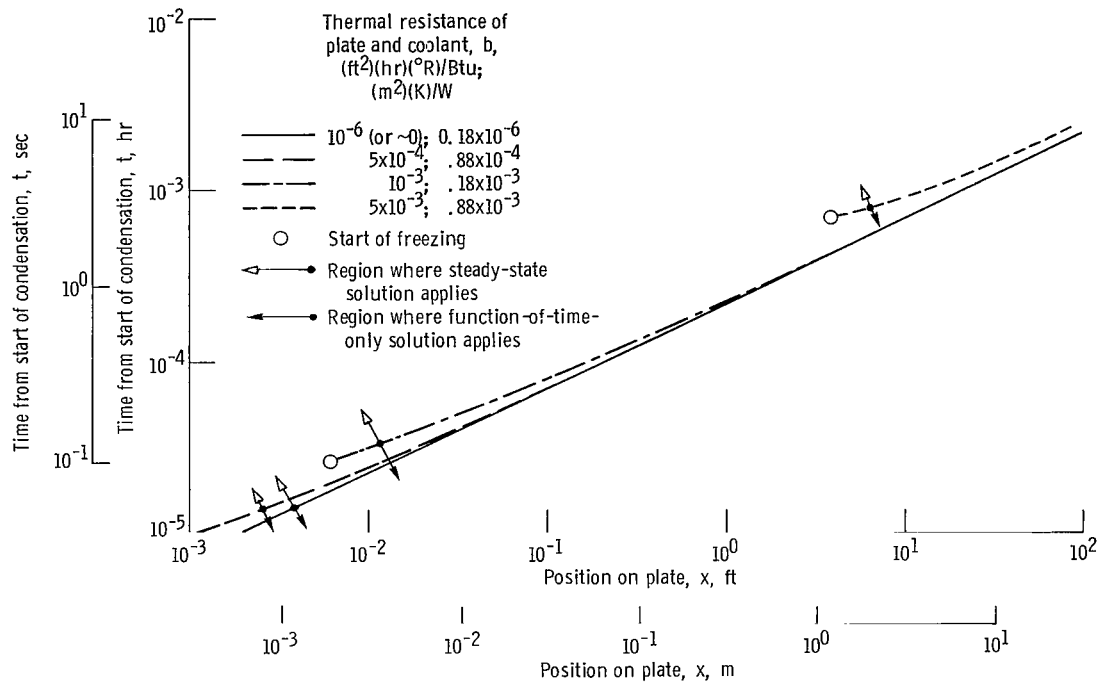
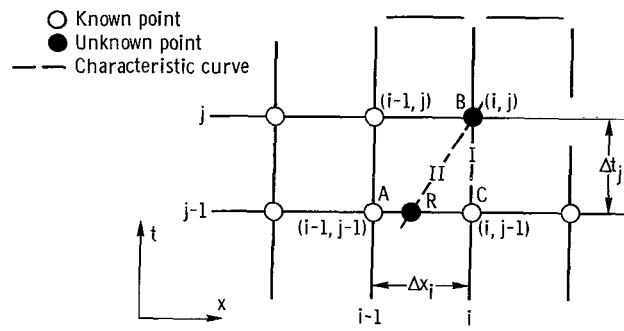
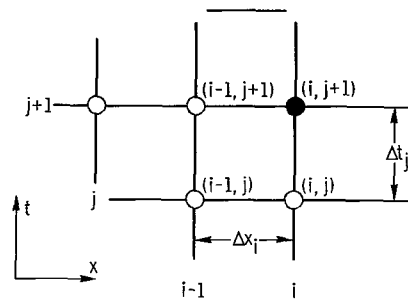


Figure 6. - Uncoupled approximation in three-phase region ( $x > X_S$ ) resulting in single characteristic curves. Subcooling parameter  $S_2 \equiv 0$ ; pressure, 14.7 psia ( $10^5 \text{ N/m}^2$ ); coolant temperature,  $140^\circ \text{ R}$  ( $77 \text{ K}$ ).



(a) Characteristic calculation grid.



(b) Direct finite difference calculation grid.

Figure 7. - Numerical calculation grids.

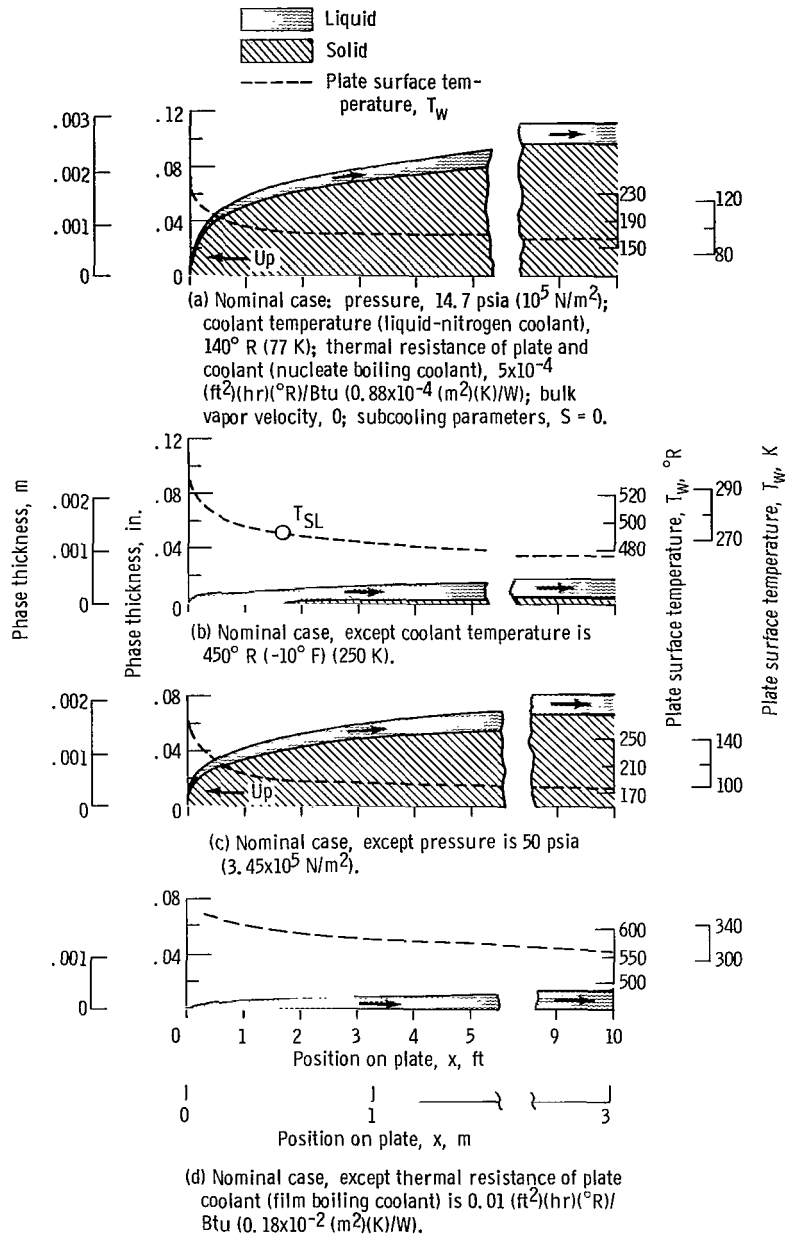


Figure 8. - Steady-state condensing and freezing of pure water vapor on cold vertical plate for a nominal case and variations.

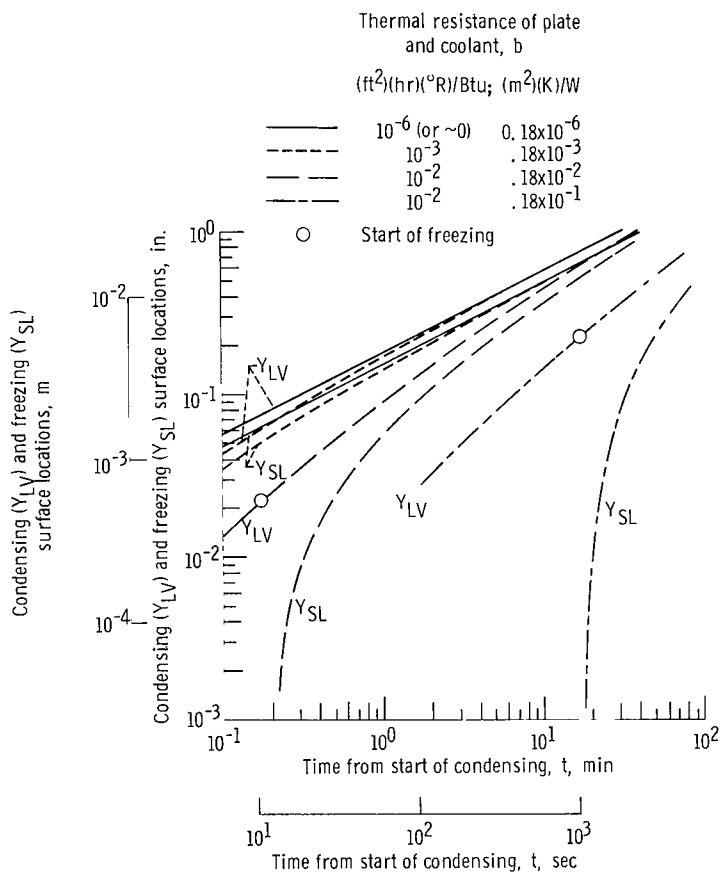


Figure 10. - Phase layer growth history as water vapor condenses and freezes on initially dry horizontal cold plate. Pressure, 14.7 psia ( $10^5 \text{ N/m}^2$ ); coolant temperature,  $140^\circ \text{R}$  ( $77 \text{ K}$ ).

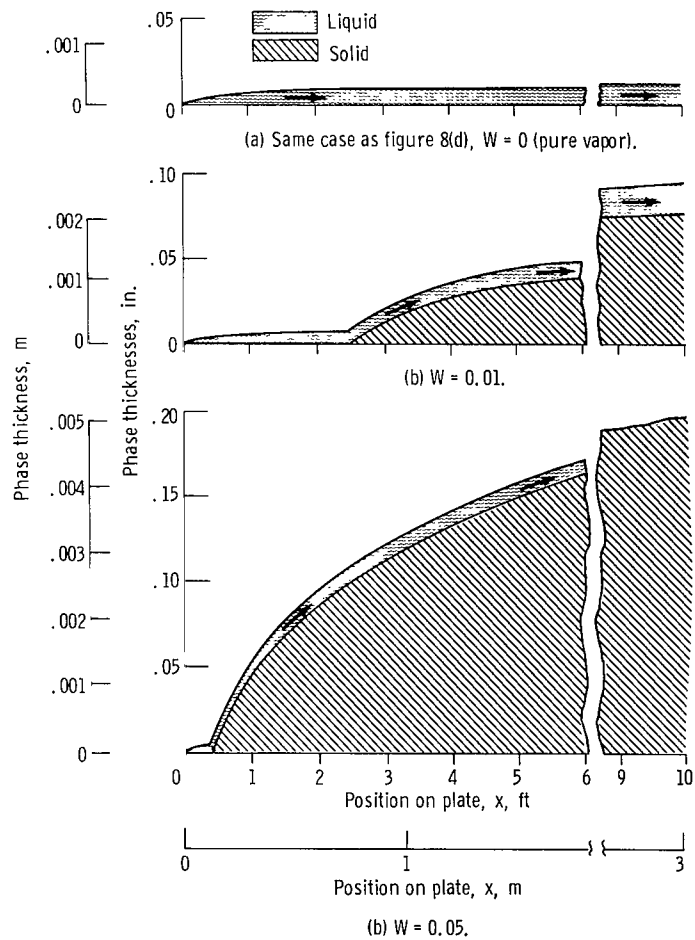


Figure 9. - Effect of noncondensable gas (air) on steady-state condensing and freezing. Same conditions as in figure 8(d) but for various noncondensable gas mass fractions  $W$ .

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